International Journal of Theoretical Physics, Vol. 40, No. 12, December 2001 (© 2001)

Renormalization and Short Distance Singular Structure

Mario Castagnino[1]

Received March 10, 2001

The relation between renormalization and short distance singular divergencies in quantum field theory is studied. As a consequence a finite theory is presented. It is shown that these divergencies originate from the multiplication of distributions (and worse-defined mathematical objects). Some of them are eliminated when the multiplication is defined based on dimensional regularization, while others disappear when the states are considered as functionals over the observables space. Nonrenormalizable theories turn to be finite, but anyhow they are endowed with infinite arbitrary constants.

1. INTRODUCTION

Quantum field theory can be reduced to the knowledge of Wightman functions (or T-ordered Feynman functions, retarded functions, or euclidean functions, etc.) (Haag, 1993; Roman, 1969). These functions are short distance singular mathematical objects (i.e. they diverge in the so-called "coincidence limits," that is, when some of their variables coincide); for example, the symmetric part of the two-point functions has a Hadamard singularity, precisely

$$w^{(2)}(x, x') = u\sigma^{-1} + v \ln|\sigma| + w \tag{1}$$

where $\sigma = (1/2)(x - x')^2$, and u, v, and w are smooth functions.[2] These local singularities give rise to the infinite ultraviolet results of quantum field theory (Brown, 1992).[3] To eliminate these infinities the theory must be renormalized in such a way that meaningless divergent expressions become meaningful. This technique is well-known but not completely satisfactory, because by using it "...we learned to peacefully coexist with alarming divergencies ... but these infinities are

[1] Instituto de Astronomía y Física del Espacio, Casilla de Correos 67, Sucursal 28, 1428 Buenos Aires, Argentina.

[2] For Wightman functions see Haag (1993), cap. VII, Eq. (3.11). For Feynman functions see De Witt (1964), Eqs. (17.61) and (16.72). For symmetrical functions see Castagnino *et al.* (1987).

[3] There is also another kind of potentially dangerous singularities as we will see in Section 6.

still with us, even though deeply buried in the formalism" (Roman, 1969). On the other hand, as we know that the short distance singularities are the cause of renormalization, if we somehow remove these singularities we will directly obtain a finite and exact quantum field theory from scratch. Phrased in another way: in this paper we will find the short distance singularity in two quantum field theory models and we will show that if these singularities are subtracted the theory turns out to be finite. The subtraction of short distance singularities has been essentially used for many years, e.g., in quantum field theory in curved space–time, (Anderson *et al.*, 2000; Birrell and Davies, 1982; Castagnino *et al.*, 1987) (and other chapters of quantum field theory, e.g., Brown, 1992, Chap. 5), but it was not considered as a general method with a rational motivation, as we are now trying to prove.

We hope that the study of the singular short distance structure will lead us, in the future, either to find Lagrangians free of this sickness (may be superstring or membrane Lagrangians) or to find more elaborated ways to remove this structure. Moreover, since the quantum field theory equations can be highly nonlinear it will be clear that, in a general case, the singular structure cannot be just removed by adding terms similar to those of the bare Lagrangian. The mechanism must be more general. Here we are presenting the physical basis of this mechanism. Essentially we believe that, since the origin of the problem is the short distance singularities, philosophically it is wrong to modify the Lagrangian until it yields a finite theory. The cure must be provided where the sickness is located.

We will find the singular structure using usual dimensional regularization (Bollini and Giambiaggi, 1972) and, in the cases where possible, Hadamard regularization (Castagnino *et al.*, 1987); moreover, we will remove it by *two different ways* at two different levels of comprehension, which we will discuss below.

1.1. Simple Subtraction Method: Detection of the Local Singularities

In Sections 2–5 we will review this well-known method with three purposes: (i) To introduce the main equations, (ii) to detect the local singularities (as in Eqs. (28), (43), (80), (104), and (116)), and (iii) to show the modification of the roles played by the coupling constant when we go from the usual method to the new one and to obtain renormalization group equations with the new method. We will study the theory in a space of n dimensions. Generically the theory will be finite for $n \neq 4$, but it will present short range singularities when $n \to 4$. For example, any two-point function will have the structure

$$w^{(2)}(x - x') = w^{(2)(s)}(x - x') + w^{(2)(r)}(x - x') \tag{2}$$

where $w^{(2)(s)}(x - x')$ is the singular component (in a sense that we will precisely describe below), that diverges when $n \to 4$ or $x \to x'$, and $w^{(2)(r)}(x - x')$ is the regular one. The subtraction method, for these functions, consists in making the singular part equal to zero or in subtracting the singular part from $w^{(2)}(x \to x')$. We will give two examples of this procedure

1. Scalar quantum field theory in a curved space–time (a theory invariant under the group of general coordinates transformations, with no self-interaction and therefore with linear equations with variable coefficients) in Section 2. In this case we only need two-point functions like those of Eq. (2).
2. $\lambda\phi^4$ theory (a theory invariant under the Poincaré group with self-interaction and therefore with nonlinear equations with constant coefficients) studied in Sections 3, 4, and 5. In the second example we will need N-point functions.

These examples are chosen not only because they are the simplest but also because the two theories are quite different and cover a large range of phenomena.[4]

Then, let us exactly state how we will define the singular and the regular components in the general case of N-point functions, in complete agreement with the usual procedures of dimensional regularization. If $w^{(N)}(x_1, x_2, \ldots, x_N)$ are some (symmetric) N-point functions (like Feynman or Euclidean functions) we can define the corresponding functional generator (Haag, 1993, Eq. (II.2.21); Brown, 1992, Eq. (3.2.11)) as

$$Z[\rho] = \exp i \left\{ \frac{1}{N!} \sum_{N=0}^{\infty} \int w^{(N)}(x_1, x_2, \ldots, x_N) \right.$$
$$\left. \times\, \rho(x_1)\rho(x_2)\cdots\rho(x_N)\, dx_1\, dx_2 \cdots dx_N \right\} \tag{3}$$

where[5]

$$w^{(N)}(x_1, x_2, \ldots, X_N) \sim \langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_N)|0\rangle \tag{4}$$

But, in a realistic field theory (namely a theory with interaction) these functions are badly defined (as the two-point function of Eq. (2)) since they are objects with mathematical properties that are *worse than those of the distributions*; moreover, if these objects were distributions all the integrals

$$\int w^{(N)}(x_1, x_2, \ldots, x_N)\rho(x_1)\rho(x_2)\cdots\rho(x_N)\, dx_1\, dx_2 \cdots\, dx_N$$

[4] For example, conformal or trace anomaly, conservation of the energy momentum tensor, etc. (see example i).

[5] The symbol $\sim$ means that the r.h.s. of the next equation can also be truncated (Haag, 1993, Eqs. (II.2.18) and (II.2.23).

would be well-defined (if, e.g., $\rho(x) \in \mathcal{S}$ the Schwarz space). But this is not the case, as we will see. Hence, $Z[\rho]$ and its derivatives are not well-defined.[6]

As we have already said in the case of quantum filed theory in curved space–time we only deal with two-point functions. But for the $\lambda\phi^4$ theory we will deal with the two-, four-, and six-point functions, in the coincidence limit where some points go to 0 and some points go to an arbitrary value z, because these are the only relevant functions in the perturbation expansion of this theory up to λ^2 order. So we will be only interested in defining the singular and regular parts of the functions $w^{(2)}(x_1, x_2)$, in the coincidence limit $x_1 = x_2 = 0$, function $w^{(4)}(x_1, x_2, x_3, x_4)$, in the coincidence limit $x_1 = x_2 = 0$, $x_3 = x_4 = z$, and function $w^{(6)}(x_1, x_2, x_3, x_4, x_5, x_6)$, in the coincidence limit $x_1 = x_2 = x_3 = 0$, $x_4 = x_5 = x_6 = z$. We will see that these coincidence limits have the general form $[w^{(2)}(0)]^\beta[w^{(2)}(z)]^\alpha$, namely the product of the power of an infinite quantity multiplied by the power of a distribution (or a worse mathematical object). In fact, these powers appear in the higher order point functions (see Haag, 1993, Eq. (II.2.18)). So we have two problems that we will solve using dimensional regularization:

i. To obtain the regular part of $w^{(2)}(0)$: It is an easy problem since via dimensional regularization, $w^{(2)}(0)$ reads

$$w^{(2)}(0) = \sum_{\gamma=0}^{C} \frac{d^{(\gamma)}}{(n-4)^\gamma} \tag{5}$$

where C is a natural number and $d^{(\gamma)}$ are some coefficients. Then the singular and regular components will be defined as

$$\left[w^{(2)}(0)\right]^{(s)} = \sum_{\gamma=1}^{C} \frac{d^{(\gamma)}}{(n-4)^\gamma} \tag{6}$$

and

$$\left[w^{(2)}(z)\right]^{(r)} = d^{(0)} \tag{7}$$

Then the regular part of $[w^{(2)}(0)]^\beta$ is simply $[d^{(0)}]^\beta$.

ii. To obtain the regular part of $[w^{(2)}(z)]^\alpha$: This is a more difficult problem since we must multiply the ill-defined function $w^{(2)}(x_1, x_2)$ by itself. But function $w^{(2)}(x_1, x_2)$ is worse than a distribution, so it cannot be multiplied by itself in a unique and well-defined way.[7] Thus we will be forced

[6] Namely, axiom B of (Haag, 1993, p. 58) is only valid for free theories, since from this axiom and Schwartz "nuclear theorem" it is shown that (4) is a distribution. Moreover, it is necessary not only that $Z[\rho]$ be well defined but also its ∂/∂_ρ-derivatives. So all $w^{(N)}(x_1, x_2, \ldots, x_N)$ must be well-defined functions after renormalization.

[7] This is where one type of divergency is "deeply buried in the formalism" (Roman, 1969). We will find another type of potentially dangerous divergencies in Section 6.

to define the multiplication procedure for, for example, $[w^{(2)}]^2$ and $[w^{(2)}]^3$ in an adhoc way based on *dimensional regularization* (see Brown, 1992, pp. 162–167, 207–214). To stress this fact we will respectively call them $[w^{(2)}]^{(d)2}$ and $[w^{(2)}]^{(d)3}$ (where the superscript "d" comes from "dimensional regularization"). Then the *multiplication procedure* will be the following:

a. Using dimensional regularization we will find that the powers are regular when $n \neq 4$, but when $n \to 4$ they behave as

$$\left[w^{(2)}(z)\right]^{(d)\alpha} = \sum_{\delta=0}^{D} \frac{d^{(\alpha,\delta)}(z)}{(n-4)^{\delta}} \tag{8}$$

where D is a natural number and $d^{(\alpha,\delta)}(z)$ are distributions (showing that, in effect, the objects we are dealing with are worse than distributions).

b. The singular and regular components will be defined as

$$\left[w^{(2)}(z)\right]^{(d)\alpha(s)} = \sum_{\delta=1}^{D} \frac{d^{(\alpha,\delta)}(z)}{(n-4)^{\delta}} \tag{9}$$

$$\left[w^{(2)}(z)\right]^{(d)\alpha(r)} = d^{(\alpha,0)}(z) \tag{10}$$

Moreover, the multiplication (ii) and the procedure to take the regular part for $z = 0$ (i) are *not commutative*. After these definitions we can substitute $[w^{(2)}(0)]^{\beta}$ and $[w^{(2)}(z)]^{\alpha}$ by $[w^{(2)}(0)^{(r)}]^{\beta}$ and $[w^{(2)}(z)]^{(d)\alpha(r)}$. Then if we consider only these regular parts, which are in general distributions (but they are regular functions in the two examples below), the functional generator $Z[\rho]$ and its derivatives (Eq. (3)) turn out to be as well-defined as the theory that it generates. The existence of singularities like those of the above equations is proved by the examples below (see also Section 5). The decompositions (6), (7) and (9), (10) are not unique, since $\infty = \infty + c$ or $\infty = c \cdot \infty$, for any finite c. This ambiguity will be present in our method, as in ordinary renormalization theory, and it yields the running coupling constants and the renormalization group, as we will see.

1.2. Functional Method

In Section 6 we will present a mathematical structure that naturally yields the elimination of the singularities. We will follow the line of thought of Laura (1997, 1998) and Castagnino and Laura (2000), where a formalism to deal with systems with continuous spectrum was introduced. It proves to be useful in the study of decay, equilibrium, and decoherence (where we have defined a final intrinsically consistent set of histories). So we claim that perhaps it is a *general formalism* that can also be used in the problem dealt with in this paper. This mathematical structure would also be the rational justification of the somehow dictatorial or childish subtraction method. This is the main contribution of the paper. The idea is the following: Coarse-graining is a well-known technique where some features of a system are considered relevant while others are not.[8] The functional method of Laura (1997, 1998) and Castagnino and Laura (2000) is a generalization of coarse-graining,[9] where the states are considered as functional over a certain space of observables.[10] Using this philosophy we will postulate that physical observables are such that they cannot see the singular components of the states because these components are irrelevant for these observables. Symmetrically, singularities could be contained in the observables and we can postulate that physical states cannot see the singular part of the observables.[11] In this way we will obtain the automatic subtraction of all kinds of singularities. There is a good physical reason for this postulate: the singularities (either of states or observables) are just mathematical artifacts originated in the oversimplified Lagrangian that we usually choose. Then, clearly physical observables or states cannot see these mathematical, unphysical objects. In a more intuitive language, the physical observables or states do not see the singularities because they are too small (pointlike). Possibly the physical observables and states just see up to Planck's length.[12]

Using the Jaynes philosophy (Jaynes, 1957a,b; Katz, 1967) we can say that if *physical* observables cannot see *mathematical* singularities (which in fact is a very reasonable position) then the (singular) states of the usual theory are in reality biased objects because they contain *arbitrary, unphysical information* (i.e., the singularities) that cannot be measured by the physical apparatuses that we have in our laboratory, i.e., our physical observables (and in reality this is an experimental fact: since apparatuses measure the values given by the finite renormalized theory). Then the (rough material) singular states, observables, and the mean values obtained from them are *biased, overinformed* objects containing dubious information, because in fact "we have a basic ignorance of the nature of infinite energies or infinitesimal distances" (Brown, 1992, p. 63), while renormalized (or free of any kind of singularities) states, observables, and mean values are *unbiased* objects containing just the physical information available. In fact, to suppose that we know

[8] Or, in observables language, the observables of theory measure only the relevant features.

[9] For example, classically, coarse graining is that particular case where the functionals are built using the characteristic functions of lattices in phase space (see Laura, 1997, 1998).

[10] Moreover, this is the natural way to face the problem since the observables are more primitive than the states; see Haag, 1993.

[11] Really this will be the case since observables are products like $\phi(x_1)\phi(x_2)\cdots$ of field $\phi(x)$, which are distributions or worse-defined mathematical objects.

[12] We could as well postulate that the singular part of the observables only sees the singular part of the state. Even if there are physical reasons to introduce this postulate in the case of decoherence, they are absent in the case of renormalization (see Section 6.2)

and measure everything would be an "inexcusable hubris" (Brown, 1992, p. 64). Moreover the resulting theory turns out to be insensitive to our degree of knowledge (originated more or less in the precision of our measurement apparatuses), thus we simply *postulate that this degree of knowledge is, and cannot be, infinite*. All this philosophy is embodied in the mathematical structure studied in Section 6.

We will discuss our conclusions in Section 7.

2. FIRST METHOD: SCALAR QUANTUM FIELD THEORY IN CURVED SPACE–TIME

This theory is the simplest nontrivial example of the method, the theory of a scalar neutral massive field in a curved space–time (of dimension n, since we need a formalism prepared for dimensional regularization) with metric $g_{\mu\nu}(x)$.[13] Let us consider the action[14]

$$(6.9) \quad S = S_{\mathrm{g}} + S_{\mathrm{m}} \tag{11}$$

where

$$(6.11) \quad S_{\mathrm{g}} = \int (-g)^{\frac{1}{2}} (16\pi G_0)^{-1} (R - 2\Lambda_0)\, d^n x \tag{12}$$

and

$$S_{\mathrm{m}} = \int (-g)^{\frac{1}{2}} L_{\mathrm{m}}\, d^n x \tag{13}$$

where L_{m} is the matter Lagrangian

$$(3.24) \quad L_{\mathrm{m}}(x) = \frac{1}{2}\{g^{\mu\nu}(x)\phi_{,\mu}(x)\phi_{,\nu}(x) - [m^2 + \xi R(x)]\phi^2\} \tag{14}$$

G_0 and Λ_0 are the bare Newton and cosmological constants respectively, m is the scalar field mass, $g^{\mu\nu}$ the inverse metric tensor (signature $+, -, -, -$), g its determinant, ξ a numerical factor, and $R(x)$ the Ricci scalar. For an in–out scattering we can define the functional generator $Z[\rho]$ such that

$$(6.15) \quad Z[0] = \langle out, 0 \mid in, 0\rangle = e^{iW} \tag{15}$$

so:

$$(6.19) \quad W = -i \ln\langle out, 0 \mid in, 0\rangle \tag{16}$$

[13] The expert reader may go directly to Section 5 and consider Sections 2–4 as a *didactical appendix* to be read after Section 7. But we consider that this didactical discussion is essential in order to convince the standard reader that the new formalism also works in practice.

[14] For the sake of conciseness we do not demonstrate the basic equations of quantum field theory in curved space–time. We just quote the number of the equation of Birrell and Davies (1982) at the beginning of each of these. In Sections 3, 4, and 5 we will use Brown (1992) for the same purpose in the $\lambda\phi^4$ case.

Then W can be computed using the effective Lagrangian L_{eff} defined by

$$(6.36) \quad W = \int [-g(x)]^{\frac{1}{2}} L_{\text{eff}}(x)\, d^n(x) \tag{17}$$

where L_{eff} reads

$$(6.37) \quad L_{\text{eff}}(x) = \frac{i}{2} \lim_{x' \to x} \int_{m^2}^{\infty} dm^2\, \Delta_F^{DS}(x, x') \tag{18}$$

where $\Delta_F^{DS}(x, x')$ is the De-Witt–Schwinger–Feynman–Green function:

$$\begin{aligned}(3.138) \quad \Delta_F^{DS}(x, x') = -i\,\Delta^{\frac{1}{2}}(x, x')(4\pi)^{-\frac{n}{2}} \int_0^{\infty} i\,ds(is)^{-\frac{n}{2}} \\ \times \exp\left[-im^2 s + \frac{\sigma}{2is}\right] F(x, x'; is)\end{aligned} \tag{19}$$

The $\sigma(x, x')$ is half the square of the geodesic distance between x and x', $\Delta(x, x')$, is the van Vleck–Morette determinant, and

$$(3.137) \quad F(x, x'; is) = a_0(x, x') + a_1(x, x')is + a_2(x, x')(is)^2 + \cdots \tag{20}$$

where the a coefficients can be obtained from Birrell and Davies (1982), Eqs. (3.131), (3.132), and (3.133), and corresponds to an expansion in the metric $g_{\mu\nu}(x)$ and its derivatives, precisely to orders 0, 2, 4, ... in these derivatives. The coefficients are biscalars, namely all the formalism is covariant under general coordinates transformation.

Equation (18) is the simple nontrivial example of the relation between L_{eff} and the two-point function $\Delta_F^{DS}(x, x')$ in the limit $x \to x'$, where in fact $\Delta^{DS}(x, x')$ has a short distance singularity that makes L_{eff} a divergent quantity, as we will see. If we want to retain the $n = 4$ dimension of L_{eff}, (length)$^{-4}$, also when $n \neq 4$, we must introduce an arbitrary mass μ. Then L_{eff} reads

$$(6.45) \quad L_{\text{eff}} = \frac{1}{2}(4\pi)^{-\frac{n}{2}} \left(\frac{m}{\mu}\right)^{n-4} \sum_{j=0}^{\infty} a_j(x) m^{4-2j}\, \Gamma\left(j - \frac{n}{2}\right) \tag{21}$$

where $a_j(x) = a_j(x, x)$ are functions of the curvatures and its derivatives, and the Γ function diverges when $n \to 4$.

2.1. Renormalization Using Dimensional Regularization

By the dimensional regularization method everything is now prepared to renormalize the theory. When $n \to 4$ the first three terms (those that correspond to orders 0, 2, and 4) diverge and we obtain the divergent or singular component of L_{eff} that reads (we have dropped the $O(n-4)$ terms):

$$(6.44)\quad L^{(s)}(x) = -(4\pi)^{-\frac{n}{2}}\left\{\frac{1}{n-4} + \frac{1}{2}\left[\gamma + \ln\left(\frac{m^2}{\mu^2}\right)\right]\right\} \times \left(\frac{4m^4 a_0(x)}{n(n-2)} - \frac{2m^2 a_1(x)}{n-2} + a_2(x)\right) \tag{22}$$

where:

$$(6.46)\quad a_0(x) = 1$$

$$(6.47)\quad a_1(x) = \left(\frac{1}{6} - \xi\right)R$$

$$(6.48)\quad a_2(x) = \frac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{1}{180}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{6}\left(\frac{1}{5} - \xi\right)\Box R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2 \tag{23}$$

where $R_{\alpha\beta\gamma\delta}$ is the curvature tensor and $R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$. The usual renormalization procedure is to absorb this singular component in the bare S_g, so we can renormalize G_0 and Λ_0 as

$$(6.50)\quad \Lambda_{\text{phys}} = \Lambda_0 + \frac{32\pi m^2 G_0}{(4\pi)^{\frac{n}{2}} n(n-2)}\left\{\frac{1}{n-4} + \frac{1}{2}\left[\gamma + \ln\left(\frac{m^2}{\mu^2}\right)\right]\right\} \tag{24}$$

$$(6.51)\quad G_{\text{phys}} = G_0/1 + 16G_0\frac{2m^2\left(\frac{1}{6} - \xi\right)}{(4\pi)^{\frac{n}{2}}(n-2)}\left\{\frac{1}{n-4} + \frac{1}{2}\left[\gamma + \ln\left(\frac{m^2}{\mu^2}\right)\right]\right\} \tag{25}$$

(where we have neglected the square terms in the bare constants) so we choose G_0 and Λ_0 in such a way that G_{phys} and Λ_{phys} turn out to be finite when $n = 4$. But this is not enough since the divergence of the $a_2(x)$ term cannot be eliminated in this way, so the theory with action S_g is not renormalizable. But, if we add three "H" terms to the gravitational Lagrangian, that are linear combinations of the three terms of Eq. (23), i.e., linear combinations of R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, and $\Box R$, (there are only three "H" terms because there is a relation among the last four terms) and renormalize the three corresponding coefficients (known as α, β, γ) the theory becomes renormalizable and finite (see Birrell and Davies, 1982, Eqs. (6.52)–(6.56)). So, from now on we will consider that these "H" terms are added to the gravitational Lagrangian (12).

But let us observe that essentially what we have done with this standard renormalization recipe is to define, as proved in Birrell and Davies (1982), a regular-subtracted Lagrangian,[15] that for $n = 4$ reads:

$$(6.59) \quad L^{(r)} = L_{\text{eff}} - L^{(s)} = \frac{1}{32\pi^2} \int_0^\infty \sum_{j=3}^\infty a_j(x)(is)^{j-3}\, e^{-im^2 s} i\, ds \tag{26}$$

which turns out to be finite and can be used instead of the divergent L_{eff}.[16] Thus we can foresee that both the standard renormalization recipe and the subtraction recipe coincide. What we have really made is a subtraction using dimensional regularization. Making the same subtraction in $\Delta_{\text{F}}^{\text{DS}}(x, x')$ (Eq. (20)), we obtain the regular $\Delta_{\text{F}}^{\text{DS}(r)}(x, x')$.[17] We will make this calculation in the next section using Hadamard regularization (Castagnino, Harari, and Nuñez, 1987) because using this method we can better show the presence and nature of the local singularities.

2.2. Hadamard Regularization and the Subtraction Recipe

Let us now see how we can directly work in the $n = 4$ case. The divergencies now appear when $x \to x'$ (not when $n \to 4$ as in the previous section). In this section we will see how the two singular behaviors are related. The effective Lagrangian (21) reads

$$(6.38) \quad L_{\text{eff}} = -\lim_{x' \to x} \frac{\Delta^{\frac{1}{2}}(x, x')}{32\pi^2} \int_0^\infty \frac{ds}{s^3}\, e^{-(m^2 s - \frac{\sigma}{2s})}[a_0(x, x') + a_1(x, x')is + a_2(x, x')(is)^2 + \cdots] \tag{27}$$

From Eqs. (19) we may compute

$$\Delta_{\text{F}}^{\text{DS}}(x, x') = -i \frac{\Delta^{\frac{1}{2}}(x, x')}{(4\pi^2)} \int_0^\infty i\, ds\, (is)^{-2}\, e^{-(m^2 s - \frac{\sigma}{2s})}[a_0(x, x') + a_1(x, x')is + a_2(x, x')(is)^2 + \cdots] = \overline{\Delta_{\text{F}}^{\text{DS}}(x, x')} + \frac{1}{2} i\, \Delta_{\text{F}}^{\text{DS}(1)}(x, x') \tag{28}$$

[15] We are using a particular criterion to define the singular component. This criterion is neither unique nor irrelevant (Castagnino *et al.*, 1986). It is clear that the singular term must have the form $\infty \times$ *geometrical object* (namely invariant under general coordinates transformations). But this object can be chosen in a variety of ways, since, as we have said, we know that $\infty = \infty + c$ or $\infty = c \cdot \infty$ for any finite c.

[16] Equation (26) already shows in Birrell and Davies (1982) that subtraction was used in quantum field theory in curved space–time, as we have in the introduction.

[17] Since this is the only nonvanishing truncated point function in the theory, all ordinary point functions of the theory are finite and they can directly be computed.

where (De Witt, 1964, Eqs. (17.61) and (17.62))

$$\overline{\Delta_{\mathrm{F}}^{\mathrm{DS}}(x,x')} = \frac{\Delta^{\frac{1}{2}}a_0}{8\pi}\delta(\sigma) - \frac{\Delta^{\frac{1}{2}}}{8\pi}\theta(\sigma)\left[\frac{1}{2}(m^2a_0 - a_1) - \frac{2\sigma}{2^2\cdot 4}(m^4a_0 - 2m^2a_1 + 2a_2) + \frac{(2\sigma)^2}{2^2\cdot 4^2\cdot 6}(m^6a_0 - 3m^4a_1 + 6m^2a_2 - 6a_3) + \cdots\right] \tag{29}$$

and

$$\Delta_{\mathrm{F}}^{\mathrm{DS(1)}}(x,x') = -\frac{\Delta^{\frac{1}{2}}a_0}{4\pi^2\sigma} + \frac{\Delta^{\frac{1}{2}}}{2\pi^2}\log\frac{e^{\gamma}}{2}|2m^2\sigma|\left[\frac{1}{2}(m^2a_0 - a_1) - \frac{2\sigma}{2^2\cdot 4}(m^4a_0 - 2m^2a_1 + a_2)\right] - \frac{\Delta^{\frac{1}{2}}}{2\pi^2}\left[\frac{1}{4}m^2a_0 - \frac{2\sigma}{2^2\cdot 4}\times\left(\frac{5}{4}m^4 - 2m^2a_1 - a_2\right) + \frac{(2\sigma)^2}{2^2\cdot 4^2\cdot 6}\left(\frac{5}{3}m^6a_0 - \frac{9}{2}m^4a_1 + \frac{15}{2}m^2a_2 - \frac{9}{2}a_3\right) + \cdots\right]\frac{\Delta^{\frac{1}{2}}}{2\pi^2}\left[\left(\frac{a_2}{4m^2} + \frac{a_3}{4m^4} + \frac{a_4}{8m^6} + \cdots\right) - \frac{2\sigma}{2^2\cdot 4}\left(\frac{a_3}{m^2} + \frac{a_4}{m^4} + \cdots\right) + \cdots\right] \tag{30}$$

According to dimensional regularization the singular part of $\Delta_{\mathrm{F}}^{\mathrm{DS}}(x,x')$ corresponds to the one with coefficients a_0, a_1, a_2 (see (22)). The remaining terms are the regular part (see (26)). Then,

$$\Delta_{\mathrm{F}}^{\mathrm{DS}(s)}(x,x') = \frac{\Delta^{\frac{1}{2}}a_0}{8\pi}\delta(\sigma) - \frac{\Delta^{\frac{1}{2}}}{8\pi}\theta(\sigma)\left[\frac{1}{2}(m^2a_0 - a_1) - \frac{2\sigma}{2^2\cdot 4}(m^4a_0 - 2m^2a_1 + 2a_2)\right] + \frac{i}{2}\left\{\frac{\Delta^{\frac{1}{2}}a_0}{4\pi^2\sigma} + \frac{\Delta^{\frac{1}{2}}}{2\pi^2}\log\frac{e^{\gamma}}{2}|2m^2\sigma|\left[\frac{1}{2}(m^2a_0 - a_1) - \frac{2\sigma}{2^2\cdot 4}(m^4a_0 - 2m^2a_1 + a_2\right] - \frac{\Delta^{\frac{1}{2}}}{2\pi^2}\left[\frac{1}{4}m^2a_0 - \frac{2\sigma}{2^2\cdot 4}\times\left(\frac{5}{4}m^4 - 2m^2a_1 - a_2\right) + \frac{(2\sigma)^2}{2^2\cdot 4^2\cdot 6}\left(\frac{5}{3}m^6a_0 - \frac{9}{2}m^4a_1 + \frac{15}{2}m^2a_2\right) + \cdots\right] + \frac{\Delta^{\frac{1}{2}}}{2\pi^2}\frac{a_2}{4m^2}\right\}. \tag{31}$$

This $\Delta_F^{DS(s)}(x, x')$ contains all the terms that diverge when $\sigma \to 0$ (like $\delta(\sigma)$, $1/\sigma$, $\log \sigma$) plus the terms with a divergent first derivative when $\sigma \to 0$ (like $\theta(\sigma)$, $\sigma\theta(\sigma)$, $\sigma \log \sigma$) plus some convergent terms when $\sigma \to 0$ (like 1, σ, σ^2). In this way we arrive at the first important conclusion of this section: *The poles of* $\Gamma(j - \frac{n}{4})$ *from which the three coefficients* a_0, a_1, *and* a_2 *originate correspond to the divergent terms or the terms with divergent derivative when* $\sigma \to 0$. There also are convergent terms in $\Delta_F^{DS(s)}(x, x')$ but they are physically irrelevant as we will soon see. The regular part of $\Delta_F^{DS}(x, x')$ reads

$$\Delta_F^{DS(r)}(x, x') = -\frac{\Delta^{\frac{1}{2}}}{8\pi}\theta(\sigma)\left[\frac{(2\sigma)^2}{2^2 \cdot 4^2 \cdot 6}(-6a_3) + \cdots\right] + \frac{i}{2}\left\{\frac{\Delta^{\frac{1}{2}}}{2\pi^2} \log \frac{e^\gamma}{2}|2m^2\sigma| \times [\sigma^2 + \cdots] - \frac{\Delta^{\frac{1}{2}}}{2\pi^2}\left[\frac{(2\sigma)^2}{2^2 \cdot 4^2 \cdot 6}\left(-\frac{9}{2}a_3\right)\right] + \frac{\Delta^{\frac{1}{2}}}{2\pi^2} \times \left[\left(\frac{a_3}{4m^4} + \frac{a_4}{8m^6} + \cdots\right) - \frac{2\sigma}{2^2 \cdot 4}\left(\frac{a_3}{m^2} + \frac{a_4}{m^4} + \cdots\right) + \right]\right\} \quad (32)$$

and contains terms that are convergent and with first derivative also convergent when $\sigma \to 0$.

Then we can define the "Hadamard regularization" as the prescription in that the singular part of $\Delta_F^{DS}(x, x')$ contains all the terms divergent or with first derivative divergent when $\sigma \to 0$ while the regular part of $\Delta_F^{DS(r)}(x, x')$ contains the terms that are convergent and with convergent first derivative when $\sigma \to 0$. At first sight the dimensional regularization and the Hadamard regularization do not coincide, since in $\Delta_F^{DS(s)}(x, x')$ there are convergent terms with all their derivatives, namely those like 1, σ, and σ^2. Nevertheless the difference is physically irrelevant since these terms are multiplied by terms $a_0(x, x')$, $a_1(x, x')$, and $a_2(x, x')$ that when $\sigma \to 0$ have the limits

$$\lim_{x' \to x} a_i(x, x') = a_i(x), \quad i = 1, 2, 3 \qquad (33)$$

From Eq. (23) we see that these terms are proportional to the linear combinations of I, R, R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, and $\Box R$, contained among the terms of the gravitational Lagrangian. Therefore the terms we are discussing can be absorbed in the gravitational action S_g supplemented by the H terms. Then, in order to unify the two regularizations, $\Delta_F^{DS(r)}(x, x')$ must be difined modulo some terms with arbitrary coefficients corresponding to the undefined terms 1, σ, and σ^2. In the effective Lagrangian these terms will produce finite terms that can be added to Λ, G, α, β, and γ (Birrell and Davies, 1982, Eq. (6.60)). The coefficients of these terms will be called l, g, a, b, and c. Dropping these terms for the moment, we can compute the regular Lagrangian corresponding to $\Delta_{\rm F}^{{\rm DS}(r)}(x,x')$ that in the coincidence limit reads

$$\lim_{x\to x'} \Delta_{\rm F}^{{\rm DS}(r)}(x,x') = \lim_{x'\to x} \frac{i\Delta^{\frac{1}{2}}}{4\pi^2}\left[\frac{a_3}{4m^4}+\frac{a_4}{8m^6}+\cdots\right] \tag{34}$$

Then as $\lim_{x'\to x}\Delta = 1$ (De Witt, 1964, Eq. (17.86)) we have

$$\lim_{x\to x'} \Delta_{\rm F}^{{\rm DS}(r)}(x,x') = \frac{i}{4\pi^2}\left\{\frac{a_3}{4m^4}+\frac{a_4}{8m^6}+\cdots\right\} \tag{35}$$

We may now add the arbitrary coefficients and obtain

$$\lim_{x\to x'} \Delta_{\rm F}^{{\rm DS}(r)} = \frac{i}{(4\pi)^2}\left\{4lm^2+ga_1+\frac{a_2}{m^2}+\frac{a_3}{m^4}+\cdots\right\} \tag{36}$$

where l and g are the already defind arbitrary coefficients and those a, b, and c correcponding to α, β, and γ are hidden in a_2. Using Eq. (18) we obtain[18]

$$L^{(r)}(x) = \frac{1}{32\pi^2}\left[2lm^4+gm^2a_1+a_2\,\log m^2+\frac{a_3}{m^2}+\cdots\right] \tag{37}$$

which turns out to be equal to Eq. (26) (except that in the quoted equation the first three terms are missing, since they are absorbed in $S_{\rm g}$ supplemented by the "H" terms), showing the coincidence of the two methods.

Therefore the subtracted $S^{(r)}$ reads

$$\begin{aligned} S^{(r)} = \int (-g)^{\frac{1}{2}}\Bigg[&-\frac{2\Lambda_0}{16\pi^2G_0}+\frac{m^4l}{16\pi^2}+\frac{R}{16\pi^2G_0}+\frac{1}{6}\frac{gm^2R}{32\pi^2} \\ &+\frac{\log m^2a_2}{32\pi^2}+\frac{a_3}{32\pi^2m^2}+\cdots\Bigg] \end{aligned} \tag{38}$$

where the quantities $-2\Lambda_0/16\pi^2G_0+m^4l/16\pi^2$ and $1/16\pi^2G_0+(1/6)(gm^2/32\pi^2)$ must be determined by physical measurements (as the α, β, and γ that are hidden in a_2).

So using Hadamard regularization and the subtraction recipe the result is, somehow, simpler since Eqs. (24) and (25) just read

$$G_{\rm phys} = G_0/1+\frac{1}{6}G_0gm^2, \qquad \Lambda_{\rm phys} = \Lambda_0-\frac{1}{2}G_0m^2l \tag{39}$$

so the bare constants are finite and would coincide, from the very beginning, with the physical ones for the choice $l=g=0$ of the arbitrary coefficients l and g. Thus using Hadamard regularization and the subtraction recipe, "*we must remove* $\Delta^{(s)}$

[18] In the first two terms, instead of $\int_{m^2}^{\infty}$ we use $-\int_0^{m^2}$ and in the third term $-\int_1^{m^2}$, because they work in these terms as $\int_{m^2}^{\infty}$ in the rest of the terms (see Birrell and Davies, 1982, p. 157).

from Δ *and use* $\Delta^{(r)}$" we have obtained the same result as in Section 2.1: all the infinities are removed and substituted by finite quantities. Thus the "subtraction recipe" works as the standard renormalization. The new recipe just consists in the elimination of the singular (or with singular first derivative) short distance components of the two-point function $\Delta_F^{DS}(x, x')$, the only relevant truncated two-point function in this theory. If we would have a $\lambda\phi^4$ interaction, more truncated point functions must be subtracted, as we will see in the next example.

3. FIRST METHOD: $\lambda\phi^4$ THEORY IN THE LOWEST ORDER

In this section we will use the subtraction method in the $\lambda\phi^4$ theory with Lagrangian[19]

$$(3.3.1) \quad L = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \Lambda \tag{40}$$

Dimensional regularization and minimal subtraction will be done following Brown (1992).

It must be clear that, as we will isolate the divergent parts and then subtract them, the theory will necessarily turn out to be finite. Thus our only aim, in Sections 3, 4, and 5 is to detect the local divergencies and to compare our method with the usual one to see how the results are obtained and to show that they are similar, (so in each paragraph "i" we will see how we can find the singular and regular parts of the objects appearing in the theory, in "ii" we will review the usual renormalization but by using our notation, and in "iii" we will see how the subtraction recipe handles the divergence problem and then we will compare the results).

3.1. Singular and Regular Parts of $\Delta_E(0)$ and Mass Renormalization

i. From Eq. (1) we know that $\Delta_E(x)$ is one of the main characters of the play. It is divergent when $x \to 0$. So we will define the singular and the regular parts of $\Delta_E(0)$, first using dimensional diagonalization and then the Hadamard one.[20] In n dimensions it reads as (just computing the

[19] In Sections 3, 4, and 5 the numbers before the equations correspond to Brown (1992) (we use the formalism and methods of this book as a sample of the standard theory). Moreover, comparing Eqs. (14) with (40) we see that there is a change of convention in the sign of the norm, so in the following sections we change this convention in order that our equations would coincide with those of the corresponding references. Also, in order to comply with Brown (1992) we will sometimes use $\Delta_F(x)$ and sometimes $\Delta_E(x)$.

[20] In both cases the singular component will have the form $\infty \times$ *geometrical object* (in this case invariant under a Lorentz transformation). Of course there are many possible subtractions, as in the previous section. In section 3.2 we will use the minimal one as in Brown (1992). In section 3.3 we will use the Hadamard one and we will show the finite difference between the two choices.

tadpole graph and neglecting nonconnected graphs that will be taken into account in Section 3.2 and 4.2)

$$(4.3.8)\quad \lim_{x\to x'} \Delta_E(x-x') = \Delta_E(0) = \frac{m^2}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\Gamma\left(1-\frac{n}{2}\right) \tag{41}$$

where μ is an arbitrary mass. We can now difine $\Delta_E^{(s)}(0)$, the divergent component of $\Delta_E(0)$. As the $\Gamma\left(1-\frac{n}{2}\right)$ behaves as

$$\Gamma\left(1-\frac{n}{2}\right) \approx \frac{2}{n-4} + \gamma \tag{42}$$

when $n \to 4$, (where $\gamma = \pi^2/12$ is the Euler–Mascheroni constant), using the minimal subtraction we find the singular part of $\Delta_E(0)$:

$$\Delta_E^{(s)}(0) = \frac{2m^2}{(4\pi)^2}\frac{1}{n-4} \tag{43}$$

In this way we have detected the local divergency. So we reach a decomposition (as (6) and (7))

$$\begin{aligned}\Delta_E(0) &= \Delta_E^{(r)}(0) + \Delta_E^{(s)}(0)\\ &= \frac{m^2}{(4\pi)^2}\left[\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\left(1-\frac{n}{2}\right) - \frac{2}{n-4}\right] + \frac{2m^2}{(4\pi)^2}\frac{1}{n-4}\end{aligned} \tag{44}$$

Then,

$$\Delta_E^{(r)}(0) = \frac{m^2}{(4\pi)^2}\left[\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\left(1-\frac{n}{2}\right) - \frac{2}{n-4}\right] \tag{45}$$

Precisely when $n \to 4$ we have

$$\Delta_E^{(r)}(0) = \frac{m^2}{(4\pi)^2}\left[\log\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma - 1\right] \tag{46}$$

where μ is the arbitrary mass, so essentially $\Delta_E^{(r)}(0)$ has an arbitrary value.

ii. Let us now see how $\Delta_E^{(r)}(0)$ is related with the mass renormalization. To correct the divergency of $\langle\phi_0(x)\phi_0(x')\rangle$ we must correct the divergency of its Fourier transform:

$$(4.3.2)\quad G_0(p) = \frac{1}{p^2 + m_0^2 + \Sigma_0(p)} \tag{47}$$

The computation of the tadpole graph (in the first λ order) yields

$$(4.3.7) \quad \Sigma_0^{(1)}(p) = \frac{1}{2}\lambda_0 \Delta_E(0) \tag{48}$$

that makes the term $m_0^2 + \Sigma_0^{(1)}(p)$ divergent. Precisely,

$$m_0^2 + \Sigma_0^{(1)}(p) = m_0^2 + \frac{1}{2}\lambda_0\left[\Delta_E^{(r)}(0) + \Delta_E^{(s)}(0)\right] \tag{49}$$

In usual renormalization we consider that the (bare) mass m_0 is divergent. Then to compensate this divergency we define a (dressed) mass m such that

$$m_0^2 + \Sigma_0^{(1)}(p) = m^2 + \Sigma^{(1)}(p) \tag{50}$$

where both terms in the r.h.s. are finite, precisely,

$$m_0^2 = m^2\left[1 - \frac{\lambda_0}{2}\Delta_E^{(s)}(0)\right] \tag{51}$$

and

$$\Sigma^{(1)}(p) = \frac{1}{2}\lambda_0 \Delta_E^{(r)}(0) \tag{52}$$

Then the physical mass is

$$(4.3.15) \quad m_{\text{phys}}^2 = m^2 + \Sigma^{(1)}(p) \tag{53}$$

where m_{phys} is a constant while $\Sigma^{(1)}(p)$ and m^2 are finite functions of μ (cf. Eq. (46)), satisfying the renormalization group equations.

iii. Using the subtraction recipe we would directly say that in Eq. (49), in reality $\Delta_E^{(s)}(0) = 0$ and we will obtain

$$m_{\text{phys}}^2 = m_0^2 + \frac{1}{2}\lambda_0 \Delta_E^{(r)}(0) \tag{54}$$

which is equivalent to (53) and where

a. m_0 plays the role of m; it is therefore finite.

b. Since $\Delta_E^{(r)}(0)$ is a function of μ, m_0 must also be a function of μ in such a way that m_{phys}^2 turns out to be a constant. Then m_0^2 satisfies the same renormalization group equation as the m^2 of Eq. (53).

This will be a common feature of subtraction recipe for all physical constants: *there is no need to introduce a dressed quantity since the bare quantity takes its role, then the bare quantity becomes a function of μ satisfying the renormalization group equations.*

In fact, in the usual theory we obtain the renormalization group equations from

$$(5.4.1) \quad Z[m_0, \lambda_0, \ldots] = Z[\mu, m, \lambda, \ldots] = \text{const.}$$

where $m_0, \lambda_0, \ldots$ are constants and $m, \lambda, \ldots$ are finite functions of μ. In the new approach this equation is changed by

$$Z[m_{\text{phys}}, \lambda_{\text{phys}}, \ldots] = Z[\mu, m_0, \lambda_0, \ldots] = \text{const.}$$

where $m_{\text{phys}}, \lambda_{\text{phys}}, \ldots$ are constants and $m_0, \lambda_0, \ldots$ are functions of μ. These two equations are formally equal, so we will obtain the same renormalization group equation in both theories.

3.2. The Cosmological Constant and the Hadamard Regularization for $\Delta_E(0)$ in the Case $\lambda = 0$

i. Let us begin making an identification. $\Delta_E^{(r)}(0)$ in flat space–time can also be obtained in the case $n = 4$ (but using Hadamard subtraction and not minimal subtraction) making all the curvatures zero in Eq. (34), (namely making all the a zero but $a_0 = 1$) and multiplying by $-i$ (since $\Delta_F \to i\Delta_E$, Brown, 1992, p. 194). So we obtain (when $\lambda = 0$)

$$\Delta_E^{(r)}(0) = \lim_{x,x' \to 0} \Delta_E^{(r)}(x, x') = \frac{4lm^2}{(4\pi)^2} \tag{55}$$

so essentially in this case $\lim_{x,x' \to 0} \Delta_E^{(r)}(x, x')$ is just an arbitrary finite constant as in the case of (46). For the case $\lambda \neq 0$ some corrections will appear in Eq. (34) (Birrell and Davies, 1982, p. 301) but the r.h.s. of Eq. (55) will always be an arbitrary constant. The origin of this ambiguity is the usual one: an infinite quantity can only be considered modulo a finite undefined constant. So the arbitrary singularity coefficient $\lim_{x,x' \to 0} \Delta_E^{(r)}(x, x')$, defined when $n = 4$, plays the same role that μ in the case $n \neq 4$. Both parameters are related, when $\lambda = 0$, by:

$$4l = \log\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma - 1 \tag{56}$$

Thus, this preliminary consideration leads us to suppose that there must be something like a cosmological constant in $\lambda\phi^4$ theory also. In fact, in traditional quantum field theory the additional infinite term that appears, due to the addition of infinite ground energy terms $\omega/2$ can be considered as an unrenormalizable cosmological constant. This term is eliminated using normal ordering. But this renormalization is better understood introducing the just-mentioned cosmological constant (Brown, 1992, Section 4.2) that must be renormalized. Using our equation we can define a cosmological constant Λ for this flat space–time theory, if we just add to the usual Lagrangian a term Λ as we have done in Eq. (40). This term reads (see (38) and (56) in the case $n = 4$)

$$\Lambda = \frac{m^4}{16\pi^2} l = \frac{m^4}{4(4\pi)^2} \left[\log\left(\frac{m^2}{4\pi\mu^2} \right) + \gamma - 1 \right] \tag{57}$$

ii. Let us see how the renormalization method introduces the cosmological constant. When $\lambda = 0$ the vacuum-to-vacuum expectation (corresponding to the vacuum one-loop graph) reads

$$\text{(4.2.1)} \quad \langle 0+ \mid 0- \rangle = \int [d\phi] \exp\left\{ -\int \left(d_E^n x \right) \times \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m_0^2 \phi^2 - \Lambda_0 \right] \right\} \tag{58}$$

Thus

$$\text{(4.2.2)} \quad \frac{\partial}{\partial m^2} \langle 0+ \mid 0- \rangle = -\frac{1}{2} \int \left(d_E^n x \right) \langle 0+ | \phi(x)^2 | 0- \rangle = -\frac{1}{2} \langle 0+ \mid 0- \rangle \int \left(d_E^n x \right) \Delta_E(0) \tag{59}$$

and

$$\text{(4.2.4)} \quad \langle 0+ \mid 0- \rangle = \exp\left[-\frac{1}{2} \int dm^2 \int \left(d_E^n x \right) \Delta_E(0) \right] \tag{60}$$

But if $\mathcal{E}$ is the cosmological energy density of the universe we also have

$$\text{(4.2.9)} \quad \langle 0+ \mid 0- \rangle = \exp\left[-\int d_E^n x \, \mathcal{E} \right] \tag{61}$$

So

$$\mathcal{E} = \frac{1}{2} \int dm^2 \, \Delta_E(0) - \Lambda_0 \tag{62}$$

where Λ_0 can be considered an integration constant. Then from Eq. (44) we have

$$\mathcal{E} = \frac{1}{2} \int dm^2 \, \Delta_E^{(r)}(0) + \frac{1}{2} \int dm^2 \, \Delta_E^{(s)}(0) - \Lambda_0 \tag{63}$$

so we can consider that Λ_0 is infinite in such a way as to cancel the infinite in $\Delta_E^{(s)}(0)$, namely,

$$\Lambda_0 = \frac{1}{2} \int dm^2 \, \Delta_E^{(s)}(0) - \mu^{4-n} \Lambda = \frac{1}{2} \frac{m_0^2}{(4\pi)^2} \frac{1}{n-4} - \mu^{4-n} \Lambda \tag{64}$$

where Λ is the finite cosmological constant. So finally

$$\mathcal{E} = \frac{1}{2} \int dm^2 \ \Delta_E^{(r)}(0) + \mu^{n-4}\Lambda \tag{65}$$

and when $n \to 4$ we have

$$(4.2.20) \quad \mathcal{E} = \frac{1}{4} \frac{m^4}{(4\pi)^2} \left[\ln\left(\frac{m^2}{4\pi\mu^2} \right) + \gamma - \frac{3}{2} \right] - \Lambda \tag{66}$$

where $\mathcal{E}$ is finite and it is not a function of μ but Λ is a function of this mass. Using the Hadamard method of point i we can directly see these facts using Eq. (57), since the μ variation is cancelled in (66). It remains a finite constant, which is unimportant since we can add an arbitrary constant to the Lagrangian (40). As usual, the condition $\mathcal{E} = const.$ originates the renormalization group equation for Λ.

iii. Directly from (62) using subtraction recipe we would have

$$\mathcal{E} = \frac{1}{2} \int dm^2 \ \Delta_E^{(r)}(0) - \Lambda_0 \tag{67}$$

that for $n \to 4$ gives

$$\mathcal{E} = \frac{1}{4} \frac{m^4}{(4\pi)^2} \left[\log\left(\frac{m^2}{4\pi\mu^2} \right) + \gamma - 2 \right] - \Lambda_0 \tag{68}$$

namely (66) with the finite merely unimportant difference $\Delta\mathcal{E} = -m^4/16(4\pi^2)$, as already discussed, and Λ_0 playing the role of Λ. Now both terms in the r.h.s. are finite and functions of μ while $\mathcal{E} = \mathcal{E}_{\text{phys}}$ is a physical constant, yielding the renormalization group equation for Λ_0 as in the usual renormalization case.

From now on we will only use the dimensional regularization since the singular structure of the higher point function is not as well-studied as the one of the two-point function.

4. FIRST METHOD: $\lambda\phi^4$ THEORY AT SECOND PERTURBATION ORDER

4.1. Singular and Regular Parts of $[\Delta_F(z)]^{(d)2}$ and the Coupling Constant Renormalization

i. Computing the fish graph we found that the scattering amplitude T reads

$$(3.5.11) \quad T = \lambda_0 + \frac{1}{2}\lambda_0^2[F(s) + F(t) + F(u)] \tag{69}$$

where λ_0 is the coupling constant and s, t, and u the Mandelstam variables; F reads[21]

$$(3.5.13) \quad F(-P^2) = -\frac{i}{2}\int (d^4 z)\, e^{iPz} \langle 0|T\phi^2(0)\phi^2(z)|0\rangle \tag{70}$$

(as in Brown (1992) we have omitted the disconnected graphs that were taken into account in Section 3.2, and we will again consider in 4.2); $\langle 0|T\phi^2(0)\phi^2(z)|0\rangle$ is the four-point function divergent coincidence limit mentioned in Section 1.1 that we must study and subtract, precisely,

$$(3.5.9) \quad \langle 0|T\phi^2(0)\phi^2(z)|0\rangle = -2\Delta_F(z)^2 \tag{71}$$

So we see that the coincidence limit is the (undefined) product of $\Delta_F(z)$ by itself. Using dimensional regularization, as explained in Section 1.1, we define

$$\langle 0|T\phi^2(0)\phi^2(z)|0\rangle = -2\Delta_F(z)^{(d)2} \tag{72}$$

We will decompose this quantity as

$$\Delta_F(z)^{(d)2} = \Delta_F(z)^{(d)2(s)} + \Delta_F(z)^{(d)2(r)} \tag{73}$$

according to the prescription (9)–(10). Then we will obtain the regular $F^{(r)}(-P^2)$ as

$$F^{(r)}(-P^2) = i\int (d^4 z)\, e^{iPz} \Delta_F(z)^{(d)2(r)} \tag{74}$$

And, if we use this $F^{(r)}$ instead of F in Eq. (69) the physical T will turn out finite. We can directly make all the procedure on $F(-P^2)$, the Fourier transform of $\Delta_F(z)^2$. Using dimensional regularization we obtain

$$(3.5.30) \quad F(-P^2) = -\frac{\mu^{n-4}}{(4\pi)^2}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 d\alpha \times \left[\frac{m^2+\alpha(1-\alpha)P^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2} \tag{75}$$

[21] Really Eq. (3.5.13) in Brown (1992) reads

$$(3.5.13) \quad F(-P^2) = i\int (d^4 z)\, e^{iPz} \Delta_F(z)^2$$

but we must remember that $\Delta_F(z)$ is a singular function (something worse than a distribution) so $\Delta_F(z)^2$ is a meaningless expression unless a multiplication procedure would be prescribed (which is done in Eq. (3.5.14) of Brown, 1992). Moreover the decomposition (3.2.19), of the same reference, that is the base of the equation above, cannot be used when two points coincide, since this decomposition is inspired in the case when these two points are far apart, as in the definition of the truncated functions.

This equation can be considered as a way to obtain the square $\Delta_F(z)^2$, i.e., to make this square when it is possible ($n \neq 4$) and then take the limit $n \to \infty$. When $n \to 4$ it is

$$\text{(3.5.31)} \quad \Gamma\left(2 - \frac{n}{2}\right) \to \frac{2}{4-n} + f(n) \tag{76}$$

where $f(n)$ is a regular function such that $\lim_{n\to 4} f(n) = -\gamma$. So, we can find the Fourier transform of the decomposition (73):

$$F(-P^2) = \mu^{n-4}[F^{(s)}(-P^2) + F^{(r)}(-P^2)] \tag{77}$$

where the factor μ^{n-4} has been displayed to make $F^{(s)}(-P^2)$ and $F^{(r)}(-P^2)$ dimensional and where

$$F^{(s)}(-P^2) = -\frac{1}{(4\pi)^2}\frac{2}{(4-n)}\int_0^1 d\alpha = -\frac{1}{(4\pi)^2}\frac{2}{(4-n)} \tag{78}$$

and

$$\text{(3.5.33)} \quad F^{(r)}(-P^2) = -\frac{1}{(4\pi)^2}\int_0^1 d\alpha \left\{\Gamma\left(2-\frac{n}{2}\right)\right.$$
$$\left. \times \left[\frac{m^2 + \alpha(1-\alpha)P^2}{4\pi\mu^2}\right]^{\frac{n}{2}-2} + \frac{2}{n-4}\right\} \tag{79}$$

Making now the inverse Fourier transformation of Eq. (78) we have that

$$i\Delta_F(z)^{(d)2(s)} = \frac{1}{(4\pi)^2}\frac{2}{n-4}\delta(z) \tag{80}$$

which, in fact, has the form announced in Eq. (9). It is singular when $z = 0$ and it shows that only the regular part is relevant when $z \neq 0$. So we have detected the local divergency. Again, the nonuniqueness of the result is shown by the presence of μ in Eq. (77).

ii. The usual renormalization procedure would be to put the singular and regular parts in (69) to obtain

$$\text{(3.5.37)} \quad T = \lambda_0 + \lambda_0^2 \frac{\mu^{n-4}}{(4\pi)^2}\frac{3}{n-4} + \frac{\mu^{n-4}}{2}\lambda_0^2$$
$$\times [F^{(r)}(s) + F^{(r)}(t) + F^{(r)}(u)] \tag{81}$$

where the physical quantity T must be finite and μ-independent. This is achieved by introducing a renormalized λ such that

$$\text{(3.5.38)} \quad \lambda_0 = \mu^{4-n}\lambda\left(1 - \frac{3\lambda}{(4\pi)^2}\frac{1}{n-4}\right) \tag{82}$$

so that λ_0 turns out to be infinite and λ finite. Then

$$(3.5.48) \quad T = \lambda + \frac{1}{2}\lambda^2\left[F^{(r)}(s) + F^{(r)}(t) + F^{(r)}(u)\right] \tag{83}$$

where all the magnitudes are finite. As T is μ-independent we can obtain the renormalization group equation for λ.

iii. According to the subtraction method, we must make zero $\Delta_{\mathrm{F}}(z)^{(d)2(s)}$ or $F^{(s)}(-P^2)$ and then we obtain the following finite physical value of T.

$$(3.5.48) \quad T = \lambda_0 + \frac{1}{2}\lambda_0^2\left[F^{(r)}(s) + F^{(r)}(t) + F^{(r)}(u)\right] \tag{84}$$

where λ_0 is a finite quantity.

Making the limit $n \to 4$ it turns out that

$$(3.5.66) \quad F^{(r)}(s) = \frac{1}{(4\pi)^2}\left\{\log\left(\frac{m^2 e^{\gamma}}{4\pi\mu^2}\right) + \sqrt{1 - \frac{4m^2}{s}} \times \log\left[\frac{\sqrt{s - 4m^2} - \sqrt{s}}{\sqrt{s - 4m^2} + \sqrt{s}} - 2\right]\right\} \tag{85}$$

We see that with the substitution $\lambda_0 \leftrightarrow \lambda$, Eqs. (83) and (84) are the same. In the case of the subtraction method λ_0 is a finite μ function, and as T is μ-independent we can obtain the same renormalization group equation as above.

4.2. The Cosmological Corrected Constant and $[\Delta_E(0)]^2$

i. In the previous section we have neglected nonconnected terms, e.g. in Eq. (48), because the mass term was a consequence of the equation

$$(4.3.5) \quad G_0^{(1)}(x - x') = -\frac{1}{2}\lambda_0 \Delta_E(0) \int \left(d_E^n \overline{x}\right) \Delta_E(x - \overline{x})\Delta_E(x - \overline{x}) \tag{86}$$

that actually reads

$$G_0^{(1)}(x - x') = -\frac{1}{2}\lambda_0 \int \left(d_E^n \overline{x}\right) \left\{\Delta_E(0)\Delta_E(x - \overline{x})\Delta_E(x - \overline{x}) + \frac{1}{4.3}\Delta_E(x - x')[\Delta_E(0)]^2\right\} \tag{87}$$

Moreover, the cosmological constant is originated in the equation

$$(3.3.9)\quad \langle 0_+ \mid 0_-\rangle = \int [d\phi]\exp\left\{-\int L_0\left(d_E^n x\right)\right\} \times \exp\left\{-\frac{\lambda_0}{4!}\int \phi^4\left(d_E^n x\right)\right\} = \langle 0_+ \mid 0_-\rangle^{(0)} - \frac{\lambda_0}{4!}\langle 0_+|\int \phi^4\left(d_E^n x\right)|0_-\rangle^{(0)} + \cdots = \langle 0_+ \mid 0_-\rangle^{(0)} - \frac{3\lambda_0}{4!}\int\left(d_E^n \overline{x}\right)[\Delta_E(0)]^2\langle 0_+ \mid 0_-\rangle^{(0)} \tag{88}$$

which corresponds to Eq. (60) with an extra term. $\langle 0_+ \mid 0_-\rangle^{(0)}$ corresponds to the case $\lambda_0 = 0$ and the second term to the nonconnected graphs (the "eight" and the "square of the figure eight," etc.). In all these expressions $[\Delta_E(0)]^2$ appears and it must be substituted by $[\Delta_E(0)^{(r)}]^2$ according to the subtraction recipe. As $\Delta_E(0)$ is not a distribution, but just the divergent quantity (41), we must only substitute it by $\Delta_E^{(r)}(0)$, using decomposition (6) and (7), and making Eqs. (87) and (88) finite.

ii. Let us now go to the renormalization method: At order two we have

$$(4.4.5)\quad \mathcal{E} = \frac{m^n}{(4\pi)^{\frac{n}{2}}}\frac{1}{n}\Gamma\left(1-\frac{n}{2}\right) - \frac{1}{2}\mu^{n-4}\frac{m^4}{(4\pi)^4}\frac{1}{n-4} + \frac{1}{2}\mu^{n-4}\frac{\lambda m^4}{(4\pi)^2}\left[\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\left(1-\frac{n}{2}\right) - \frac{1}{n-4}\right]^2 + \frac{1}{2}\mu^{n-4}\frac{m^4}{(4\pi)^2}\frac{1}{n-4}\left(1-\frac{\lambda}{(4\pi)^2}\frac{1}{n-4}\right) - \Lambda_0 \tag{89}$$

It can be checked that the first two lines of this equation are finite when $n \to 4$. So we must define a renormalized cosmological constant Λ such that

$$(4.4.6)\quad \Lambda_0 = \mu^{n-4}\left[\frac{1}{2}\frac{m^4}{(4\pi)^2}\frac{1}{n-4}\left(1-\frac{\lambda}{(4\pi)^2}\frac{1}{n-4}\right) + \Lambda\right] \tag{90}$$

Then we have the final finite expression

$$\mathcal{E} = \frac{m^n}{(4\pi)^{\frac{n}{2}}}\frac{1}{n}\Gamma\left(1-\frac{n}{2}\right) - \mu^{n-4}\frac{1}{2}\frac{m^4}{(4\pi)^2}\frac{1}{n-4} - \mu^{n-4}\Lambda + \frac{1}{2}\mu^{n-4}\frac{\lambda m^4}{(4\pi)^2}\left[\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\frac{1}{2}\Gamma\left(1-\frac{n}{2}\right) - \frac{1}{n-4}\right]^2 \tag{91}$$

which is finite when $n \to 4$. In fact, when $\lambda = 0$, we have that

$$\mathcal{E} = \frac{m^n}{(4\pi)^{\frac{n}{2}}} \frac{1}{n} \Gamma\left(1 - \frac{n}{2}\right) - \mu^{n-4} \frac{1}{2} \frac{m_0^4}{(4\pi)^2} \frac{1}{n-4} - \mu^{n-4} \Lambda \tag{92}$$

which is a finite quantity, as we have proved in Section 3.2 (it corresponds to $\langle 0_+ \mid 0_- \rangle^{(0)}$), while

$$\left[\left(\frac{m_0^2}{4\pi \mu^2} \right)^{\frac{n}{2}-2} \frac{1}{2} \Gamma\left(1 - \frac{n}{2}\right) - \frac{1}{n-4} \right] \tag{93}$$

is finite for (45), so the r.h.s. of Eq. (91) is finite. When $n \to 4$ we find

$$\begin{aligned} \mathcal{E} = {} & \frac{m^4}{4(4\pi)^2} \left[\log\left(\frac{m^2}{4\pi \mu^2} \right) + \gamma - \frac{3}{2} \right] + \frac{\lambda}{8} \frac{m^4}{(4\pi)^2} \\ & \times \left[\log\left(\frac{m^2}{4\pi \mu^2} \right) + \gamma - \frac{3}{2} \right]^2 - \Lambda \end{aligned} \tag{94}$$

The terms of the r.h.s. are μ functions that originate the renormalization group equation as usual.

iii. Using directly the subtraction method in Eq. (89) we would have when $n \to 4$:

$$\begin{aligned} \mathcal{E} = {} & \frac{m^4}{4(4\pi)^2} \left[\log\left(\frac{m^2}{4\pi \mu^2} \right) + \gamma - 1 \right] + \frac{\lambda}{8} \frac{m^4}{(4\pi)^2} \\ & \times \left[\log\left(\frac{m^2}{4\pi \mu^2} \right) + \gamma - 2 \right]^2 - \Lambda_0 \end{aligned} \tag{95}$$

with all terms finite and Λ_0 a function of μ as usual, which is equal to (94) with the exception of the already known unimportant constant. For both methods the renormalization group equation can be obtained prescribing that $\mathcal{E}$ would not be a function of μ.

4.3. The $[\Delta_E(z)]^{(d)3}$ and the Wave Function Renormalization

i. In reality mass renormalization of Section 3.1 is based in the Green function:

$$(4.3.4) \quad G_0^{(1)}(x - x') = -\frac{1}{4!} \lambda_0 \int \left(d_E^n \overline{x} \right) \langle \phi(x) \phi(x') \phi^4(\overline{x}) \rangle^{(0)} \tag{96}$$

that can be written as

$$(4.3.5) \quad G_0^{(1)}(x - x') = -\frac{1}{4!} \lambda_0 \Delta_E(0) \int \left(d_E^n \overline{x} \right) \Delta_E(x - \overline{x}) \Delta_E(x' - \overline{x}) \tag{97}$$

In the next order we must compute

$$(4.5.2)\quad G_0^{(2)}(x-x') = \frac{1}{2}\left(-\frac{\lambda_0}{4!}\right)\int \left(d_E^n y\right)\left(d_E^n z\right) \times \langle \phi(x)\phi(x')\phi^4(y)\phi^4(z)\rangle^{(0)} \tag{98}$$

Computing this Green function, as we have done with the previous one, we find

a. Vacuum *disconnected graphs*: They are the "eight" and the "square eight," etc. which are removed by ordinary renormalization of the cosmological constant or by the corresponding subtraction that makes this constant finite but undefined, as shown in Eqs. (57) or (95), (Brown, 1992, pp. 205–206).

b. Disconnected *two legs graph*: It is the product of the "tadpole" by the "eight." Both graphs have already been considered either by renormalization or subtraction.

c. Connected *two legs graphs*: Namely:

c_1. The "double scoop" or "double bubble" graph, with an integral

$$(4.5.3)\quad \Sigma_0^{(2,1)}(p) = -\frac{1}{4}\lambda_0^2 \int \left(d_E^n y\right)\Delta_E(y)^2\Delta_E(0) \tag{99}$$

(which really is not a function of p). It has two factors.

– $\Delta_E(0)$, which was considered in Section 3.1 and made finite by both recipes.

– $\Delta_E(y)^2 = \Delta_E(y)^{(d)2}$, which was considered in Section 4.1, since the integral in Eq. (99) is just the integral in Eq. (74) with $P = 0$, which also was made finite by both recipes. So $\Sigma_0^{(2,1)}$ turns out to be finite either way. Finally let us observe that in $\Sigma_0^{(2,1)}$ the typical expression $[w^{(2)}(0)]^\beta[w^{(2)}(z)]^\alpha$, of Section 1.1, appears for the first time in its complete version.

c_2. The "setting sun" graph, which is a function of p

$$(4.5.4)\quad \Sigma_0^{(2,2)}(p) = -\frac{1}{6}\lambda_0^2 \int \left(d_E^n x\right)\Delta_E(x)^3\, e^{-ipx} \tag{100}$$

To deal with this integral we must first compute $\Delta_E^{(d)3}(x)$ multiplying $\Delta_E(x)$ three times, then make its dimensional regularization, and finally its Fourier transform $\Delta_E^{(d)3}(p)$. We obtain

$$(4.5.37)\quad \Sigma_0^{(2,2)}(p) = -\frac{1}{6}\left(\frac{\lambda}{(4\pi)^2}\right)^2 p^2\left(\frac{p^2}{4\pi\mu^2}\right)^{n-4} \times \frac{\Gamma\left(\frac{n}{2}-1\right)^3\Gamma(3-n)}{\Gamma\left(\frac{3n}{2}-3\right)} \tag{101}$$

As when $n \to 4$

$$\frac{\Gamma\left(\frac{n}{2}-1\right)^3\Gamma(3-n)}{\Gamma\left(\frac{3n}{2}-3\right)} \to \frac{1}{2}\frac{1}{(n-4)} \tag{102}$$

then

$$\left[\Sigma_0^{(2,2)}(p)\right]^{(s)} = -\frac{1}{12}\left(\frac{\lambda}{(4\pi)^2}\right)^2 p^2 \frac{1}{n-4} \tag{103}$$

Then, we conclude that:

$$\begin{aligned}\Delta_E^{(d)3(s)}(x) &= \frac{1}{2}\left(\frac{1}{2\pi}\right)^n \frac{1}{(4\pi)^2}\frac{1}{n-4}\int p^2\, e^{ipx}\left(d_E^n p\right) \\ &= \frac{1}{2}\frac{1}{(4\pi)^2}\frac{1}{n-4}\nabla^2\delta(x)\end{aligned} \tag{104}$$

which, in fact, has the form announced in Eq. (9). It is local, since it is singular when $z = 0$ and vanishing for $z \neq 0$. We have detected another local singularity. In the finite limit when $n \to 4$ we obtain

$$(4.5.38) \quad \Sigma_0^{(2,2)}(p) = -\frac{1}{12}\left[\frac{\lambda}{(4\pi)^2}\right]^2 p^2\left(\log\frac{p^2}{4\pi\mu^2} + \text{const.}\right) \tag{105}$$

Subtracting all singularities the propagator $G_0^{(2)}(x-x')$ turns out to be finite to the second λ order. But, of course, an ambiguity appears in the constant of Eq. (105) that must be fixed by a measurement.

ii. In the renormalization theory we must add all the results of the connected graphs to obtain

$$\begin{aligned}(4.5.39) \quad G_0(p)^{-1} = p^2&\left\{1 - \frac{1}{12}\left[\frac{\lambda}{(4\pi)^2}\right]^2\frac{1}{n-4}\right\} + m_0^2 \\ &+ m^2\left\{\frac{\lambda}{(4\pi)^2}\left[\frac{1}{n-4} + \text{finite}\right]\right\} - \left[\frac{\lambda}{(4\pi)^2}\right]^2 \\ &\times\left\{m^2\left[\frac{2}{(n-4)^2} + \frac{1}{2}\frac{1}{(n-4)}\right]\right. \\ &\left.+ \text{finite function of } p^2\right\}\end{aligned} \tag{106}$$

To eliminate the infinities via renormalization a renormalized $G(p)$ is defined as

$$(4.5.40) \quad G_0(p) = z_1^2 G(p) \tag{107}$$

where

$$(4.5.41) \quad z_1^2 = 1 + \frac{1}{12}\left[\frac{\lambda}{(4\pi)^2}\right]^2 \frac{1}{n-4} \tag{108}$$

then up to the order λ^2 we have

$$\begin{aligned}(4.5.42) \quad G(p)^{-1} = p^2 + m_0^2 + m^2\left\{\frac{\lambda}{(4\pi)^2}\left[\frac{1}{n-4} + \text{finite}\right]\right\} \\ -\left[\frac{\lambda}{(4\pi)^2}\right]^2 \left\{m^2\left[\frac{2}{(n-4)^2}\right.\right. \\ \left.\left.+\frac{5}{12}\frac{1}{(n-4)} + \text{finite}\right]\right\}\end{aligned} \tag{109}$$

and the renormalized mass reads

$$\begin{aligned}(4.5.43) \quad m_0^2 = m^2\left\{1 - \frac{\lambda}{(4\pi)^2}\frac{1}{(n-4)} + \left[\frac{\lambda}{(4\pi)^2}\right]^2\right. \\ \left.\times\left[\frac{2}{(n-4)^2} + \frac{5}{12}\frac{1}{(n-4)}\right]\right\}\end{aligned} \tag{110}$$

Then

$$G(p)^{-1} = p^2 + m^2 + \text{finite}(p^2) \tag{111}$$

All terms are finite and $G(p)^{-1}$, m^2, are finite(μ) are μ functions; p^2 is the physical constant quantity that originates the renormalization group. The renormalization of Eqs. (107) and (108) is usually considered as a wave function renormalization:

$$(4.5.46) \quad \phi_0(x) = z_1\phi(x) \tag{112}$$

where $\phi(x)$ is the renormalized field.

iii. Using the subtraction recipe Eq. (111) reads

$$G(p)^{-1} = p^2 + m_0^2 + \text{finite}(p^2) \tag{113}$$

since all the infinities disappear from Eqs. (109) and (110), but a finite undeterminate constant remains that must be fixed by a measurement that corresponds to one of the wave function renormalization. As usual, m_0^2 and finite(p^2) are μ functions that originate the renormalization group. There will be two equations: one for the p^2 coefficients and one for the remaining terms. Moreover, from Eq. (108) with no infinity we have

$z_1 = 1$ and there is no need of the wave function renormalization.[22] Then using our recipe, we get the same result.

5. FIRST METHOD: $\lambda\phi^4$ THEORY AT ANY ORDER; MORE GENERAL $\lambda\phi^l$ THEORIES; SPECULATIONS ON NON-RENORMALIZABLE THEORIES

We can now follow a well-known path. For the $\lambda\phi^4$ theory the superficial divergence is

$$(5.2.21) \quad D = 4 - N \tag{114}$$

where N is the number of external legs of the graph. Then, only graphs with $N = 2$ and $N = 4$ have basic divergencies. Moreover the covergence of all the graphs to λ^2 can be reduced to prove the convergence of the primitive divergent graph (Ramond, 1981, p. 144), namely the tadpole and the fish graphs, the double scoop graph, and the setting sun graph (and the nonconnected graphs), which were studied in the previous sections. These graphs are finite under ordinary renormalization (or if the subtraction recipe is used). So, repeating these calculations to any order all graphs of the renormalized theory are finite and the theory turns out to be finite to all orders (Dyson, 1949; t'Hooft and Veltman, 1972). $\lambda\phi^4$ theory can be considered as renormalizable since it has a finite number of primitive divergent graphs and therefore a finite number of relevant singular point functions, namely two. So we now know that *using subtraction method the theory is directly finite to any order*. The only difference with the counterterms formalism is that now the undetermined finite coefficients are located at the infinite local singularities. In the renormalizable case these coefficients combine among themselves in such a way so as to produce a finite number of undetermined quantities that are computed by a finite number of measurements, as in the usual theory.

To complete the panorama we can study the problem in more general scalar field theories. Theories with interactions $\lambda\phi^l$ with $l > 4$ turn out to be nonrenormalizable because they have an infinite number of primitive divergent graphs and therefore an infinite number of relevant singular point functions that cannot be compensated with the finite number of terms of the bare Lagrangian. But the subtraction recipe can anyhow be used, making all these singular functions finite, and these theories would become also finite. *So all theories can be made finite if we use the subtraction recipe*.

[22] This fact must be most welcome since both the "bare" and "renormalized" fields now satisfy the same equal time commutation relations. Moreover we can introduce a finite Z^{-1} coefficient before p^2 if we would like to have a finite wave function renormalization as in Eq. (6.4.18) (see Brown, 1992).

In fact, let us consider what we know about this kind of theories:

i. To make the theory finite we must make finite (by renormalization or subtraction) all the superficially divergent subgraphs ($D \geq 0$). The mass dimension in each term is the superficial divergency.

ii. The divergent terms are polynomials of finite order in the external momentum. Using dimensional regularization with minimal subtraction the coefficients of these polynomials are found to contain positive integer powers of the parameters of the theory multiplied by poles in $n-4$ (Brown, 1992, p. 235).[23] So the typical *divergent* term reads

$$P(p_1, p_2, \ldots, p_N) = \sum A^{\gamma}_{\alpha\beta\delta_1,\ldots,\delta_N} \frac{m^{\alpha}\lambda^{\beta}\cdots}{(n-4)^{\gamma}} p_1^{\delta_1} p_2^{\delta_2} \cdots p_N^{\delta_N} \tag{115}$$

that under a Fourier transform $w_N^{(s)}(x_1, x_2, \ldots, x_N) \approx \int dp_1 \int dp_2 \cdots \int dp_N\, P(p_1, p_2, \ldots, p_N) e^{-ix_1p_1} e^{-ix_2p_2} \cdots e^{-ix_Np_N}$ corresponds to the local singularity:

$$w_N^{(s)}(x_1, x_2, \ldots, x_N) \approx \sum A^{\gamma}_{\alpha\beta\delta_1,\ldots,\delta_N} \frac{m^{\alpha}\lambda^{\beta}\cdots}{(n-4)^{\gamma}} \times \nabla^{\delta_1}\delta(x_1)\nabla^{\delta_2}\delta(x_2)\cdots\nabla^{\delta_N}\delta(x_N) \tag{116}$$

as in Eq. (104), i.e., singularities of the (6) type. All the singularities $\nabla^{\delta i}\delta(x_i)$ are well-defined distributions in variable x_i (there are no meaningless expressions as $\delta(0)\int_0^{\infty} d\omega$ that we will consider and eliminate in the next section) multiplied by infinite poles $1/(n-4)^{\gamma}$.

So let us compare the two methods:

i. Renormalization: In this case the divergent (115) terms must be compensated by counterterms like

$$\frac{\delta m^{\alpha'}\delta\lambda^{\beta'}\cdots}{(n-4)^{\gamma}} p_1^{\delta_1} p_2^{\delta_2} \cdots p_N^{\delta_N} \tag{117}$$

where $\alpha \neq \alpha'$ and $\beta \neq \beta'$, but $\alpha + \beta = \alpha' + \beta'$ in such a way so as to have the same dimension (or the same superficial divergence D). It is clear that in general such counter terms must be infinite and will be only finite in particular cases (renormalizable theories). Moreover non-renormalizable theories are considered noncontrollable, since they must have an infinite number of counter terms, implying new interaction terms of growing power.

[23] There are also non polynomial divergencies. But they can be eliminated, in the computation of $N > 2$ functions, if we begin considering the most elementary subgraphs and make minimal subtraction in these subgraphs, to proceed recursively doing the same in more complex subgraphs and we finish subtracting the overall divergency of the graph.

ii. Divergencies will disappear using the subtraction recipe and the theory will turn out finite anyhow. In fact, as in our method the Lagrangian remains untouched, and we can make the theory finite simply subtracting the divergent terms. Then all the $\gamma > 0$ terms will disappear and $w_N^{(r)}(x_1, x_2, \ldots, x_N)$ will be a well-defined function.[24] As from the general formalism of quantum theory (Haag, 1993) we are used to deal with a host of infinite divergent point functions,[25] to deal with a similar host of finite point functions, obtained via the subtraction recipe, it cannot be a major theoretical problem. So under our method both renormalizable and non-renormalizable theories are finite. Nevertheless, in renormalizable theories the ambiguous terms are combined in such a way that the unknown parameters of the theory can be computed with a finite number of physical data, while in the case of nonrenormalizable theories this number is infinite.[26]

Then using our method, non-renormalizable theories most likely make some sense and, if they have small coupling constants, probably would yield good results, using a few terms of the perturbation expansion and a few physical data, but of course we do not know yet if they have any physical relevance. Moreover, in recent years it has become increasingly apparent that the usual renormalization is not a fundamental physical requirement (Weinberg, 1995, Vol. 1, p. 518). We stop our speculation here, since this will be the subject of forthcoming researches.

6. SECOND METHOD

In this section we will try to find a theoretical justification for the subtraction method, following the authors' of the following references: Bogolyubov *et al.* (1975), Haag (1993), Segal (1947, 1969), and van Hove (1955, 1956, 1957a,b, 1959). We will also find new potentially dangerous divergencies hidden in the formalism, which will also be eliminated. The quoted authors consider that the first object that must be taken into account in quantum field theory is the set of observables O that we will use (belonging to the space of the relevant observables $\mathcal{O}$). Then the states ρ can be considered as the functionals over these observables yielding the mean values $(\rho \mid O)$. If the spectra of the observables of the problem are discrete we have $(\rho \mid O) = \mathrm{Tr}(\rho O)$. If one or many of these spectra are continuous the problem is more difficult because the last symbol is ill-defined. This

[24] In reality we also have an infinite set of counter terms, but not in the Lagrangian; they are the singular terms of the point functions that must be subtracted from these functions to obtain the regular terms so that they are precisely located in the place where they are needed.

[25] Like those listed in footnote 2.

[26] For example, in the $\lambda\phi^4$ theory, the renormalization group shows that all the residues of the poles depend on those of the first-order poles (Brown, 1992, p. 241). Namely all the ambiguities corresponding to higher divergencies depend on the first-order ambiguities, and therefore all these ambiguities can be computed with just some measurements. In the general $\lambda\phi^l$ case there is no such miracle and we must deal with infinite ambiguities.

happens when, the energy spectrum is continuous. In Laura and Castagnino (1997, 1998) we solve this problem (based on the mathematical structure introduced in Antoniou *et al.*, 1997), finding good results for many physical problems. In the present paper we deal with short distance divergences, related with the position operators, which also have a continuous spectrum. So we will try to adapt the method of Laura and Castagnino (1997, 1998) to this new problem. But first let us review the formalism of this paper.

6.1. van Hove Formalism

Let us consider a system with a Hamiltonian H with continuous energy spectrum $0 \leq \omega < +\infty$. In the simple case at least some generalized observables read

$$O = \int\int d\omega\, d\omega' \,[O_\omega \delta(\omega - \omega') + O_{\omega\omega'}]|\omega\rangle\langle\omega'| \tag{118}$$

where O_ω and $O_{\omega\omega'}$ are regular functions (with properties we will discuss below). These observables are contained in a space $\mathcal{O}$. The introduction of distributions like $\delta(\omega - \omega')$ is necessary because the "singular term" $O_\omega\delta(\omega - \omega')$ appears in observables that cannot be left outside the space $\mathcal{O}$, like the identity operator, the Hamiltonian operator, or the operators that commute with the Hamiltonian. So, even in this simple case the observables contain δ functions (while in more elaborate cases they will also contain other kind of distributions). Symmetrically a generalized state reads

$$\rho = \int\int d\omega\, d\omega' \,[\rho_\omega \delta(\omega - \omega') + \rho_{\omega\omega'}]|\omega\rangle\langle\omega'| \tag{119}$$

where ρ_ω and $\rho_{\omega\omega'}$ also are regular functions (with properties to be defined). These states are contained in a convex set of states $\mathcal{S}$. The introduction of distributions like $\delta(\omega - \omega')$ is also necessary in this case because the "singular term" $\rho_\omega\delta(\omega - \omega')$ appears in generalized states that cannot be left outside the set $\mathcal{S}$, like the equilibrium state.[27] With this mathematical structure it is impossible to calculate something like $\mathrm{Tr}(\rho O)$ because the meaningless terms $\delta(0)\int_0^\infty d\omega$ appear. *This is the main problem* (*if* $O_\omega \neq 0$ *and* $\rho_\omega \neq 0$). Let us keep in mind that with the old philosophy we are just considering the mean value $\mathrm{Tr}(\rho O)$ as a simple *inner product* (and in doing so we have the problem of $\delta(0)\int_0^\infty d\omega$).

The problem is solved if we consider the characteristic algebra of the operators $\mathcal{A}$ (see the complete version in Castagnino and Ordoñez, 2000) containing the space of the self-adjoint observables $\mathcal{A}_s$, which contain the minimal subalgebra $\hat{\mathcal{A}}$ of the observables that commute with the Hamiltonian H (that we can consider to be the typical "diagonal" operators). Then we have

$$\hat{\mathcal{A}} \subset \mathcal{A}_S \subset \mathcal{A} \tag{120}$$

[27] Usually this state is not considered in the scattering theory. So it is only *potentially dangerous* for more general theories.

Now we can make the quotient

$$\frac{\mathcal{A}}{\hat{\mathcal{A}}} = \mathcal{V}_{\mathrm{nd}} \tag{121}$$

where $\mathcal{V}_{\mathrm{nd}}$ would represent the vector space of equivalent classes of operators that do not commute with H (the "nondiagonal operators"). These equivalence classes read

$$[a] = a + \hat{\mathcal{A}}, \quad a \in \mathcal{A} \tag{122}$$

So we can decompose $\mathcal{A}$ as

$$\mathcal{A} = \hat{\mathcal{A}} + \mathcal{V}_{\mathrm{nd}} \tag{123}$$

(this decomposition corresponds to the one in Eq. (118). But neither of the two of the last two equations is a direct sum, since we can add and subtract an arbitrary $a \in \hat{\mathcal{A}}$ from each term of the r.h.s. of the last equation.

At this point we can ask ourselves which the measurement apparatuses are that really matter in the case of decoherence under an evolution e^{-iHt}. Certainly these apparatuses are those that measure the observables that commute with H and that are contained in $\hat{\mathcal{A}}$. Therefore they correspond to diagonal matrices $\sim\delta(\omega - \omega')$. The apparatuses that measure observables that do not commute with H (that corresponds to matrices with off-diagonal terms) are contained in $\mathcal{V}_{\mathrm{nd}}$. The terms corresponding to the second kind of apparatuses (either in the observables or in the corresponding states) must vanish when $t \to \infty$, so they must be endowed with mathematical properties adequate for producing this limit. Riemann–Lebesgue theorem tells us that this fact take places if functions $O_{\omega\omega'}$ are regular (and also the $\rho_{\omega\omega'}$, see below). So we define a subalgebra of $\mathcal{A}$, that can be called a van Hove algebra, as:

$$\mathcal{A}_{\mathrm{vh}} = \hat{\mathcal{A}} \oplus \mathcal{V}_{\mathrm{r}} \subset \mathcal{A} \tag{124}$$

where the vector space $\mathcal{V}_{\mathrm{r}}$ is the space of observables with $O_\omega = 0$ and $O_{\omega\omega'}$ a regular function. Now the $\oplus$ is a direct sum because $\hat{\mathcal{A}}$ contains $\delta(\omega - \omega')$ and $\mathcal{V}_{\mathrm{r}}$ just regular functions, and a kernel cannot be both a δ and a regular function. Moreover, as our observables must be selfadjoint, the space of observables must be

$$\mathcal{O} = \mathcal{A}_{\mathrm{vhS}} = \hat{\mathcal{A}} \oplus \mathcal{V}_{\mathrm{rS}} \subset \mathcal{A}_{\mathrm{S}} \tag{125}$$

where $\mathcal{V}_{\mathrm{rS}}$ contains only self-adjoint operator (namely $O^*_{\omega\omega'} = O_{\omega'\omega}$). Restriction (125) is just the choice (coarse-graining) of the relevant measurement apparatuses for our problem, those that measure the diagonal terms in $\hat{\mathcal{A}}$ and those that measure the nondiagonal terms that vanish when $t \to \infty$ in $\mathcal{V}_{\mathrm{rS}}$. Moreover $\mathcal{O} = \mathcal{A}_{\mathrm{vhS}}$ is dense in $\mathcal{A}_{\mathrm{S}}$ (because any distribution can be approximated by regular functions) and therefore essentially it is the minimal possible coarse-graining. Let us call

$|\omega) = |\omega\rangle\langle\omega|$ the vectors of the basis of $\hat{\mathcal{A}}$ and $|\omega, \omega') = |\omega\rangle\langle\omega'|$ of those of $\mathcal{V}_{\mathrm{rS}}$. Then a generic observable of $\mathcal{O}$ reads

$$O = \int d\omega\, O_\omega |\omega) + \int\int d\omega\, d\omega'\, O_{\omega'} |\omega, \omega') \tag{126}$$

which is a vector in the basis $\{|\omega), |\omega, \omega')\}$, where O_ω and $O_{\omega\omega'}$ are regular functions (with properties exactly described in Laura and Castagnino (1997, 1998) and omitted here, as we will do with all the functions that will appear in this brief review).

The states must be considered as linear functional over the space $\mathcal{O}$ ($\mathcal{O}'$ the dual of space $\mathcal{O}$ (Bogolyubov *et al.*, 1975; Segal, 1947, 1969; van Hove, 1955, 1956, 1957a,b, 1959):

$$\mathcal{O}' = \mathcal{A}'_{\mathrm{vhS}} = \hat{\mathcal{A}}' \oplus \mathcal{V}'_{\mathrm{rS}} \subset \mathcal{A}'_{\mathrm{S}} \tag{127}$$

Therefore the states read

$$\rho = \int d\omega\, \rho_\omega (\omega| + \int\int d\omega\, d\omega'\, \rho_{\omega\omega'} (\omega, \omega'| \tag{128}$$

where ρ_ω and $\rho_{\omega\omega'}$ are regular functions and $\{(\omega|, (\omega, \omega'|\}$ is the cobasis of $\{|\omega), |\omega, \omega')\}$. The set of these generalized states is the convex set $\mathcal{S} \subset \mathcal{O}'$. Now the mean value

$$(\rho \mid O) = \int d\omega\, \rho_\omega O_\omega + \int\int d\omega\, d\omega'\, \rho_{\omega\omega'} O_{\omega'\omega} \tag{129}$$

is well-defined and yields reasonable physical results (Laura and Castagnino, 1997, 1998).[28] In the last equation terms like $\delta(0) \int_0^\infty d\omega$ have disappeared. *This is the simple trick that allows us to deal with the singularities in a rigorous mathematical way and to obtain correct physical results in Laura and Castagnino (1997, 1998) and Castagnino and Laura (2000).* Essentially we have defined a new observable space $\mathcal{O}$ that contains the observables O of Eq. (126) that are adapted to solve our problem. In this way we have found a method to deal with the singular terms containing Dirac's deltas. We are now considering the mean value $(\rho \mid O)$ not as an inner product but as the action *functional* ρ acting on the vector O (and the $\delta(0) \int_0^\infty d\omega$ have disappeared). Decoherence is a consequence of Riemann–Lebesgue theorem in the time evolution of the last equation, namely,

$$(\rho(t) \mid O) = \int d\omega\, \rho_\omega O_\omega + \int\int d\omega\, d\omega'\, e^{-i(\omega-\omega')t} \rho_{\omega\omega'} O_{\omega'\omega} \tag{130}$$

[28] Moreover, the introduction of the singular observables automatically yield the introduction of the singular states (Laura and Castagnino, 1997, 1998).

6.2. The Formalism in the Simplest Case

Let us now use the same technique to deal with the singularities of quantum field theory. But first let us remember that in quantum field theory there coexists at least two different mathematical structures:

– The abstract Hilbert space $\mathcal{H}$ where the field $\phi(x)$ is an operator and the vacuum state $|0\rangle$ a vector. The multiplication in the characteristic algebra $\mathcal{A}$ is the multiplication of these operators. This is not the place where divergencies are produced. Therefore we will not modify this structure.
– The vector space of functions of $N(N \to \infty)$ variables $x_1, x_2, \ldots, x_N$ where the functions $\phi(x_1)\phi(x_2)\cdots\phi(x_N)$ can be considered as the coordinates of the vectors of a vector space $\mathcal{N}$ in a basis $|x_1, x_2, \ldots, x_N)$. Since we have proved that really the "functions" $\phi(x_1)\phi(x_2)\cdots\phi(x_N)$ are distributions or worse, we will give to this space the mathematical structure that we explained in the previous subsection.[29]

The characteristic algebra is $\mathcal{A} = \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{N}$.

Let us begin with the case of just two variables to see the analogy with the previous section. Then, as the observables like $\phi(x)\phi(x')$ are distributions (or worse) it is reasonable to consider that all the observables are singular.[30] Let us begin with the simplest case i.e.: with just the singularity (80). Then our observables would read (like in (118) or (80) with $z = x - x'$):

$$O = \int\int dx\, dx' \left[\frac{O_x}{n-4}\delta(x-x') + O_{xx'}\right] |x, x') \tag{131}$$

where O_x and $O_{xx'}$ are regular functions. But if we continue the road of Eqs. (118) and (119) we will find the same problems as above. On the other hand using the philosophy just explained,[31] we can define the space of observables

$$\mathcal{O} = \mathcal{A}_{\text{vhS}} = \hat{\mathcal{A}} \oplus \mathcal{V}_{\text{rS}} \subset \mathcal{A}_S \tag{132}$$

where $\hat{\mathcal{A}}$ is now the space of the $\delta(x - x')$–singularity with pole $(n-4)^{-1}$ and $\mathcal{V}_{\text{nS}}$ is the space of regular observables measured by physical apparatuses. O_x and

[29] Mathematically speaking this would be the one of a "nuclear" space $\mathcal{N}$, namely the generalization of the ordinary N-rank tensor space to the case where the N indices are continuous. In the future we will base an axiomatic quantum field theory using this mathematical structure.

[30] We may say that we are using the continuous spectrum of the position operator (Aparicio *et al.*, 1995a,b; García-Álvarez and Gaioli, 1997), that is $-\infty < x < +\infty$ and define the basis $|x, x')$ as $|x\rangle\langle x|$. But this is not necessary since we can directly say that the space $\mathcal{N}$ of vectors with coordinates $\phi(x)\phi(x')$ has a basis $|x, x')$.

[31] But now referred to the measurement apparatuses, i.e., those that measure variable x, which now take the role of variable ω.

$O_{xx'}$ are regular function and $\mathcal{O} = \mathcal{A}_{\text{vhS}}$ is dense in $\mathcal{A}_{\text{S}}$. Then we may transform the Eq. (131) to make it similar to (126), namely

$$O = \int dx\, \frac{O_x}{n-4} |x) + \int\int dx\, dx'\, O_{xx'} |x, x') \tag{133}$$

so now the observables are vectors of a space $\mathcal{O} \subset \mathcal{N} \otimes \mathcal{H} \otimes \mathcal{H}$ of basis $\{|x), |x, x')\}$. Then the states of this system are just some linear functionals over the space $\mathcal{O}$.

$$\mathcal{O}' = \mathcal{A}'_{\text{vhS}} = \hat{\mathcal{A}}' \oplus \mathcal{V}'_{rS} \subset \mathcal{A}'_{S} \tag{134}$$

For a moment let us postulate that the singularities in the states also do exist.[32] In this perspective the state must be linear combinations in the basis $\{(x|, (x, x'|\}$ (where $\{(x|, (x, x'|\}$ is the cobasis of $\{|x), |x, x')\}$), so they must read

$$\rho = \int dx\, \rho_x (x| + \int\int dx\, dx'\, \rho_{xx'} (x, x'| \tag{135}$$

where ρ_x and $\rho_{xx'}$ are regular functions. With these definitions the action of functional $(\rho|$ over the vector $|O)$ reads

$$(\rho \mid O) = \int dx\, \frac{\rho_x O_x}{n-4} + \int\int dx\, dx'\, \rho_{xx'} O_{x'x} \tag{136}$$

and it will be well-defined when $n = 4$ only if the first term of the r.h.s. vanishes. But this is precisely the case since, based in the arguments of Section 1.2, we know that either the real physical observables must be such that $O_x = 0$, namely they cannot see the singularities of the states (really because they only are mathematical artifacts, etc.) or $\rho_x = 0$ (namely the states cannot see the singularities of the observables, etc.). Then either $O_x = 0$ or $\rho_x = 0$ and the last equation reads

$$(\rho \mid O) = \int\int dx\, dx'\, \rho_{xx'} O_{x'x} \tag{137}$$

and therefore we have eliminated the singular term $\rho_x O_x/(n-4)$ of Eq. (136), which now has no physical effect. *In this way we can justify the elimination of* all *singular terms as we have done with* (80) *as we will see.*[33]

[32] This is not really the case as we will see in the next subsection.

[33] Of course we can also directly say that the term $\int dx\, \rho_x O_x/(n-4)$ is unphysical. But there is a difference between Eq. (129) and the last equation. In the former the singular observables see the singular states and therefore it has two terms. In the latter there are either singular observables or singular states and they have only one term. Therefore the two coarse-graining use in Sections 6.1 and 6.2 are different. This fact is no surprising since the singular terms (in ω) are necessary in the case of decoherence to represent the diagonal final state but these singular terms (in x) must disappear in the case of quantum field theory since this is the way divergent poles disappear. The two different coarse-grainings are introduced to explain two different observed physical facts.

6.3. The Formalism in the General Case

To generalize this idea let us go back to Eq. (3). We know that the functional $Z[\rho]$ and its derivatives define the whole theory. Moreover, following the above ideas it must be written as:[34]

$$Z[\rho] = \exp i(\rho \mid O) \tag{138}$$

where

$$|O) = |\phi(x_1)\phi(x_2)\cdots\phi(x_N)) \tag{139}$$

$\phi(x_1)\phi(x_2)\cdots\phi(x_N)$ being the components of a vector $|O) \in \mathcal{A} = \mathcal{N} \otimes \mathcal{H} \otimes \mathcal{H}$ for any N and

$$(\rho| = \rho(x_1)\rho(x_2)\cdots\rho(x_N)|0\rangle\langle 0| \tag{140}$$

where $(\rho| \in \mathcal{A}' = \mathcal{N}' \otimes \mathcal{H} \otimes \mathcal{H}$. Remember that what really matters for our analysis is that "functions" $\phi(x_1)\phi(x_2)\cdots\phi(x_N)$ and $\rho(x_1)\rho(x_2)\cdots\rho(x_N)$ are in spaces $\mathcal{N}$ and $\mathcal{N}'$ while the way to operate with $|0\rangle\langle 0|$ over the field $\phi(x)$ remains the usual one since it takes place in space $\mathcal{H}$. Moreover, these are the observables and states that really matter since they define $Z[\rho]$. The observable $|O)$ is the generalized version of Eq. (133); thus

$$O = \sum_N \left[\int dx_1 \int dx_2 \cdots \int dx_N \, O^{(r)}_{x_1 x_2 \cdots x_N} |x_1, x_2, \ldots, x_N) \right.$$
$$\left. + \sum_{N,\alpha_i,i} \int dx_1 \int dx_2 \cdots \int dx_{N-i} \, \frac{O^{(\alpha_i,s)}_{N,x_1x_2\cdots x_{N-i}}}{(n-4)^{\alpha_i}} |N, \alpha_i, x_1, x_2, \ldots, x_{N-i}) \right] \tag{141}$$

for all possible N and all possible coincidence limits symbolized by i. As before we can define an observable space

$$\mathcal{O} = \mathcal{A}_{\mathrm{vhS}} = \hat{\mathcal{A}} \oplus \mathcal{V}_{\mathrm{rS}} \subset \mathcal{A}_S \tag{142}$$

where:

i. The first term in the r.h.s. of Eq. (141) belongs to the space $\mathcal{V}_{\mathrm{rS}}$, with basis $\{|x_1, x_2, \ldots, x_N)\}$ and regular functions $O^{(r)}_{x_1 x_2 \cdots x_N}$.
ii. The second term of the r.h.s. of Eq. (141) belongs to the space $\hat{\mathcal{A}}$, the algebra of the singularities of Eq. (116) with basis $\{|N, \alpha_i, x_1, x_2, \ldots, x_{N-i})\}$ and regular functions $O^{(\alpha_i,s)}_{N,x_1x_2\cdots x_{N-i}}$. Then the singular terms are like those of Eq. (116).

[34] The next symbol contains a sum over the indices $N = 0, 1, 2, \ldots$

$(\rho|$ is the generalized version of the state $\rho(x_1)\rho(x_2)\cdots\rho(x_N)|0\rangle\langle 0|$. Then, if we repeat the reasoning of Eq. (135), these generalized states would read

$$\rho = \sum_N \int dx_1 \int dx_2 \cdots \int dx_N \, \rho^{(r)}_{x_1 x_2 \cdots x_N} (x_1, x_2, \ldots, x_N| + \sum_{N,\alpha_i,i} \int dx_1 \int dx_2 \cdots \int dx_{N-i} \, \rho^{(\alpha_i,s)}_{N,x_1 x_2 \cdots x_{N-i}} (N, \alpha_i, x_1, x_2, \ldots, x_{N-i}| \tag{143}$$

As above we can defined the state space as

$$\mathcal{O}' = \mathcal{A}'_{\mathrm{vhS}} = \hat{\mathcal{A}}' \oplus \mathcal{V}'_{\mathrm{rS}} \subset \mathcal{A}'_{\mathrm{S}} \tag{144}$$

where

i. The first term of the r.h.s. of Eq. (143) belongs to the space $\mathcal{V}'_{\mathrm{rS}}$ with basis $\{(x_1, x_2, \ldots, x_N|\}$ and regular functions $\rho^{(r)}_{x_1 x_2 \cdots x_N}$.
ii. The second term of the r.h.s. of Eq. (143) belongs to the space $\hat{\mathcal{A}}$, with basis $\{(N, \alpha_i, x_1, x_2, \ldots, x_{N-i}|\}$ and regular functions $\rho^{(\alpha_i,S)}_{N,x_1 x_2 \cdots x_{N-i}}$.

Then

$$(\rho \mid O) = \sum_N \int dx_1 \int dx_2 \cdots \int dx_N \, \rho^{(r)}_{x_1 x_2 \cdots x_N} O^{(r)}_{x_1 x_2 \cdots x_N} + \sum_{N,\alpha_i,i} \int dx_1 \int dx_2 \cdots \int dx_{N-i} \, \rho^{(\alpha_i,s)}_{N,x_1 x_2 \cdots x_{N-i}} O^{(\alpha_i,s)}_{N,x_1 x_2 \cdots x_{N-i}} (n-4)^{-\alpha_i} \tag{145}$$

which is a mathematically well-defined object only when $n \to 4$, if only the coordinates $\rho^{(r)}_{x_1 x_2 \cdots x_N}$ and $O^{(r)}_{x_1 x_2 \cdots x_N}$ do not vanish. But this is the case since *either*

i. the physical observables in reality read

$$O = \sum_N \left[\int dx_1 \int dx_2 \cdots \int dx_N \, O^{(r)}_{x_1 x_2 \cdots x_N} |x_1, x_2, \ldots, x_N) \right] \tag{146}$$

since they have only the regular part (because they do not see the singularities of the states, etc.) so they have no singular $(n-4)^{-\alpha_i}$ terms *or*

ii. the states in reality read

$$\rho = \sum_N \left[\int dx_1 \int dx_2 \cdots \int dx_N \, \rho^{(r)}_{x_1 x_2 \cdots x_N} (x_1, x_2, \ldots, x_N| \right] \tag{147}$$

since they have only the regular part (because they do not see the singularities of the observables, etc.) so they have no singular $(n-4)^{-\alpha_i}$ terms. But here we have a better argument: they have only a regular part since the functions $\rho(x_1)\rho(x_2)\ldots\rho(x_N)$ of Eq. (140) *are usually considered regular, with no singularity.*

Therefore if we use the functional idea embodied in Eq. (145), or better Eq. (140), and the regular state of Eq. (146) or regular observables in (147) we just have

$$Z[\rho] = \exp i \sum_N \int dx_1 \int dx_2 \cdots \int dx_N \, O^{(r)}_{x_1 x_2 \cdots x_N} \rho^{(r)}_{x_1 x_2 \cdots x_N} \tag{148}$$

which is finite, and the same happens with the $\partial/\partial\rho$ derivatives of $Z[\rho]$. Thus the theory is finite. *So the theory becomes finite just supposing that the physical observables are regular (namely, just using as observables the real physical apparatuses in our laboratory that give us* finite measurements*) or the functions* $\rho(x_1)\rho(x_2)\cdots\rho(x_N)$ *are regular (*which is the usual supposition*), and adopting the* functional *approach based in the ideas of* Bogolyubov *et al.* (1975), Haag (1993), Segal (1947, 1969), and van Hove (1955, 1956, 1957a,b, 1959). In this way the subtraction method is justified. Instead if we use the naive usual formalism where all the characters belong to Hilbert spaces and are multiplied using the ordinary *inner product,* $Z[\rho]$ will be singular and the theory must be renormalized.

7. CONCLUSION

Sometimes renormalization is considered as a *miracle* (Brown, 1992, p. 243; Ramond, 1981, p. 172). In fact, there is an infinite bare mass m_0 (which being infinite can hardly be considered as "bare"), and an infinite counterterm; that plus the bare mass gives the finite physical "dressed" mass m (which being finite is less dressed than the bare one); there is an infinite bare coupling constant and a counterterm such that the subtraction of all these infinities gives the right answer. This is a *pure miracle*![35]

Now let us consider the same phenomenon according to the ideas in this paper: We have chosen the simplest Lorentz-invariant lagrangian L, constructed using a scalar filed ϕ, to base our theory. It is too much to assume that L would give us the right answers both for long and short distances. In fact, it works remarkably well for long distances but it behaves badly for short ones, since it produces short distance singularities in the relevant N-point functions. So let us eliminate these singularities and we will obtain both the correct short and long distance behavior. This is the best we can do with Lagrangian L and the best we have until more refined Lagrangians will be invented (using perhaps superstrings, membranes, etc.). Moreover, the singular structure is pointlike and a pure mathematical artifact, and therefore undetectable by the measurement apparatuses, so it must be eliminated, in some way or other. So there is no miracle in the finite nature of the theory

[35] The author himself confesses that it was really difficult to understand and to teach this miracle.

and there is a logical explanation of what really is going on. All these facts are embodied in the rigorous mathematical structure of Section 6.

Only a *minor miracle* remains. The numerical constant of some (renormalizable) models are determined by a finite number of measurements, while others (unrenormalizable) need an infinite number. Really it is a very small miracle compared with the former one. We are used to deal with systems that can be defined with a finite number of parameters (e.g., mechanical systems) while others have an infinite number (e.g., the initial conditions of classical electromagnetic fields or mechanical systems with an infinite number of parameters like fluid with variable density or viscosity). Then what really remains is a very big practical problem: how to work and solve quantum field systems similar to the second kind.[36] We do not propose a solution but we believe that we have thrown light upon the real nature of the problem.

ACKNOWLEDGMENT

This paper was partially supported by Grants PID-0150 of CONICET (Argentina National Research Council) and Ex-198 of Buenos Aires University.

REFERENCES

Anderson, P. R., Molina-Paris, C., and Ramsay, S. (2000). *Physical Review D* **61**, 126561.

Antoniou, I., Laura, R., Suchanecki, Z., and Tasaki, S. (1997). *Physica A* **241**, 737.

Aparicio, J. P., Gaioli, F. H., and García-Álvarez, E. T. (1995a). *Physical Review A* **51**, 96.

Aparicio, J. P., Gaioli, F. H., and García-Álvarez, E. T. (1995b). *Physics Letters A* **200**, 233.

Birrell, N. D. and Davies, P. C. (1982). *Quantum Field Theory in Curved Space*, Cambridge University Press, Cambridge.

Bogolyubov, N. N., Logunov, A. A., and Todorov, I. J. (1975). *Introduction to Axiomatic Quantum Field Theory*, Benjamin, London.

Bollini, G. and Giambiaggi, J. J. (1972). *Physics Letters* **40B**, 566.

Brown, L. (1992). *Quantum Field Theory*, Cambridge University Press, Cambridge.

Castagnino, M., Gunzig, E., Nardone, P., and Paz, J. P. (1986). *Physical Review D* **34**, 3698.

Castagnino, M., Harari, D., and Nuñez, C. (1987). *Journal of Mathematical Physics* **28**, 184.

Castagnino, M. and Laura, R. (2000). *Physical Review A* **62**, 022107.

Castagnino, M. and Ordonež, A. (2000). *An Algebraic Formalization of Quantum Decoherence*. Unpublished.

De Witt, B. (1964). Dynamical theory of groups and fields. In *Relativity, Groups, and Topology*, C. De Witt and B. De Witt, eds., Gordon & Breach, New York.

Dyson, F. J. (1949). *Physical Review* **75**, 1736.

García-Álvarez, E. T. and Gaioli, E. T. (1997). *International Journal of Theoretical Physics* **36**, 2391.

Haag, R. (1993). *Local Quantum Physics*, Springer, Berlin.

Jaynes, E. T. (1957a). *Physical Review* **106**, 620.

[36] Maybe superstrings or membranes are bookkeeping devices that allow us to tame an infinite number of data as a function $y = f(x)$ encompassing an infinite number of data: the infinite relations between each value of the variable x with each value of the variable y.

Jaynes, E. T. (1957b). *Physical Review* **108**, 171.
Katz, A. (1967). *Principles of Statistical Mechanics, the Information Theory Approach*, Freeman, San Francisco.
Laura, R. and Castagnino, M. (1997). *Physical Review A* **57**, 4140.
Laura, R. and Castagnino, M. (1998). *Physical Review E* **57**, 3948.
Ramond, P. (1981). *Field Theory: A Modern Primer*, Benjamin, London.
Roman, P. (1969). *Introduction to Quantum Theory*, Wiley, New York.
Segal, I. E. (1947). *Annals of Mathematics* **48**, 930.
Segal, I. E. (1969). *Bulletin of the American Mathematical Society* **75**, 1390.
t'Hooft, G. and Veltman, M. (1972). *Nuclear Physics B* **44**, 189.
van Hove, L. (1955). *Physica* (*Amsterdam*) **21**, 901.
van Hove, L. (1956). *Physica* (*Amsterdam*) **22**, 343.
van Hove, L. (1957a). *Physica* (*Amsterdam*) **23**, 268.
van Hove, L. (1957b). *Physica* (*Amsterdam*) **23**, 441.
van Hove, L. (1959). *Physica* (*Amsterdam*) **25**, 268.
Weinberg, S. (1995). *The Quantum Theory of Fields*, Cambridge University Press, Cambridge.